I) Baroclinic instability of a two-layer fluid

a) Write down the equations for conservation of quasigeostrophic potential vorticity expressed in terms of an upper layer and lower layer streamfunction ($\psi_1$, $\psi_2$) and the interface height $\eta$ ($h_1 = H - \eta$, $h_2 = H + \eta$). Assume for simplicity that the Coriolis term, $f$, is a constant, and that the mean thicknesses of the two layers are both $H$.

\[
\begin{align*}
\frac{Dq_1}{Dt} &= \frac{D}{Dt} (\ ? ) = 0 \\
\frac{Dq_2}{Dt} &= \frac{D}{Dt} (\ ? ) = 0
\end{align*}
\]  

(1a) (1b)

b) Linearize ($\frac{Dq}{Dt} \approx \frac{D}{Dt} q' + \frac{D'}{Dt} \bar{q} = 0$) your equations about the mean zonal velocities in the two layers ($U$, $-U$) (thus, $\psi_1 = -Uy + \psi_1'$ and $\psi_2 = Uy + \psi_2'$). You may want to use the hydrostatic relationship $p_2 = p_1 + (\rho_2 - \rho_1)g\eta$ to express the interface height in terms of the streamfunction difference ($\psi_1 - \psi_2$).

e) Assume solutions in the upper and lower layer of the form $\psi_j = \bar{\psi}_j \sin(l y) e^{i(kx-\omega t)}$, $j = 1, 2$ where $l$ is chosen to match the boundary conditions at the northern and southern edges of the channel ($l = n\pi/L$, $n = 1, 2, 3, \ldots$). Solve for frequency $\omega$ and find the conditions under which $\omega$ is imaginary, thus indicating exponential growth. How does this vary with $k$ and $U$? Does this make sense?
d) Briefly (1 paragraph) describe the role (if any) that baroclinic instability plays in the general circulation of the atmosphere.

2) (H#8.12) Derive the phase speed \( c \) for the Eady wave

\[
c = \frac{\Lambda H}{2} \pm \frac{\Lambda H}{2} \left[ 1 - \frac{4 \cosh \alpha H}{\alpha H \sinh \alpha H} + \frac{4}{\alpha^2 H^2} \right]^{1/2}
\]

3) (H#8.13) Unstable baroclinic waves play an important role in the global heat budget by transferring heat poleward. Show that for the Eady wave solution the poleward heat flux averaged over a wavelength,

\[
\bar{v'T'} = \frac{1}{L} \int_0^L v'T' \, dx'
\]

is independent of height and is positive for a growing wave. How does the magnitude of the heat flux at a given instant change if the mean wind shear is doubled?

4) Construct a four-box energy diagram for both the atmosphere and the ocean (zonal mean KE and PE, and eddy KE and PE for the atmosphere; time-mean KE and PE, and eddy KE and PE for the ocean). Indicate the direction and rate of flow of energy as well as the magnitude per m^2 of each type of energy. This information is in Peixoto and Oort’s *Physics of Climate* and elsewhere. In one paragraph discuss the role of flow instabilities in the energy conversions.