Methods for solving the RTE with scattering

Approximate methods:
Single scattering approximation
2-stream approximation
Eddington approximation

Accurate methods:
Successive orders of scattering
Spherical harmonics
Adding-doubling
Discrete ordinates

2-stream approximation

Starting point: radiative transfer equation:

$$\mu \frac{dI}{d\tau} = -I + (1-a)B + \underbrace{\frac{a}{4\pi} P(\Omega_0, \Omega) \cdot F^s \cdot e^{-\frac{\tau}{\mu_0}}}_{S} + \frac{a}{4\pi} \int_{4\pi} P(\Omega', \Omega) \cdot I(\Omega') \cdot d\Omega$$

Meaning of terms?

Two-stream method is based on 5 approximations

***Approximation 1*:** Only sunlight is considered, no thermal emission.

So the RTE becomes: $$\mu \frac{dI}{d\tau} = -I + S + \frac{a}{4\pi} \int_{4\pi} P(\Omega', \Omega) \cdot I(\Omega') \cdot d\Omega$$

Approximation #2

Only azimuthally averaged intensity is considered: $I(u) = \frac{1}{2\pi}\int_0^{2\pi} I(u,\varphi)\cdot d\varphi$

Method good and not good for what quantities and applications?

Effect of approximation #2 on the phase function:

After azimuthal information is lost by averaging, the scattering angle and phase function cannot depend on azimuth, only on zenith angle. $\cos\Theta = \Omega \bullet \Omega' = \Omega_x\Omega'_x + \Omega_y\Omega'_y + \Omega_z\Omega'_z \Longrightarrow P(\cos\Theta) = P(u,u')$

The azimuthally averaged phase function is: $\frac{1}{2\pi}\int_0^{2\pi} P(u,u',\varphi-\varphi')\cdot d(\varphi-\varphi') = \frac{1}{2\pi}P(u,u')\int_0^{2\pi} d(\varphi-\varphi') = P(u,u')$

Effect of approx. #2 on Radiative Transfer Equation (*RTE*): $u\frac{dI}{d\tau} = -I + S + \frac{a}{4\pi}\int_{4\pi} P(\Omega',\Omega)\cdot I(\Omega')\cdot d\Omega$

Because of the azimuthal symmetry, the last term is

$$\frac{a}{4\pi}\int_{4\pi} P(\Omega',\Omega)\cdot I(\Omega')\cdot d\Omega = \frac{a}{4\pi}\int_{-1}^{1}\int_0^{2\pi} P(uu')\cdot I(u')\cdot d\varphi\cdot du = \frac{a}{4\pi}\int_{-1}^{1} P(uu')\cdot I(u')\int_0^{2\pi} d\varphi\cdot du =$$

$$= \frac{a}{4\pi}\int_{-1}^{1} P(uu')\cdot I(u')\cdot 2\pi\cdot du = \frac{a}{2}\int_{-1}^{1} P(uu')\cdot I(u')\cdot du$$

So the RTE becomes $u\frac{dI}{d\tau} = -I + S + \frac{a}{2}\int_{-1}^{1} P(uu')\cdot I(u')\cdot du'$

Approximation #3

Phase function is expanded in two Legendre polynomial terms:

Earlier, we saw that $$P(\cos\Theta) \approx \sum_{l=0}^{2N-1} (2l+1)\cdot \chi_l \cdot P_l(\cos\Theta)$$

where P_l is l^{th} order Legendre polynomial. and χ_l is case specific Legendre coefficient,

In our case $\cos\Theta = uu'$, and l=0,1, and we use that:

$$P_0(x) = 1 \qquad P_1(x) = x$$

$$\chi_0 = 1 \qquad \chi_1 = \frac{1}{2}\int_{-1}^{1} P(\cos\Theta)\cdot\cos\Theta\cdot d(\cos\Theta) = g$$

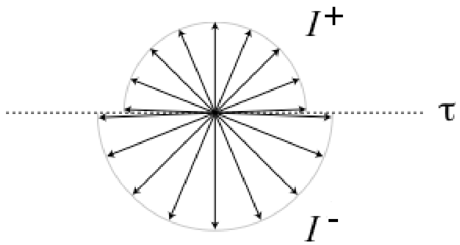
This gives: $$P(uu') \approx \sum_{l=0}^{1} (2l+1)\cdot \chi_l \cdot P_l(uu') = 1 + 3guu'$$

So the RTE becomes $$u\frac{dI}{d\tau} = -I + S + \frac{a}{2}\int_{-1}^{1} (1 + 3guu')\cdot I(u')\cdot du'$$

Approximation #4

Radiance field is simple: upward and downward radiances are isotropic, but have different values at I^+ and I^-:

$$I(u) = \begin{cases} I^-, \text{ if } u \geq 0 \\ I+, \text{ if } u < 0 \end{cases}$$



So the RTE becomes a set of two equation:

$$u\frac{dI^-}{d\tau} = -I^- + S^- + \frac{a}{2}\int_0^1 (1+3guu')\cdot I^- \cdot du' + \frac{a}{2}\int_{-1}^0 (1+3guu')\cdot I^+ \cdot du'$$

$$-u\frac{dI^+}{d\tau} = -I^+ + S^+ + \frac{a}{2}\int_0^1 (1+3guu')\cdot I^- \cdot du' + \frac{a}{2}\int_{-1}^0 (1+3guu')\cdot I^+ \cdot du'$$

Approximation #5

Use an effective $\bar{u}$ value in the radiative transfer equation

$$\bar{u}\frac{dI^-}{d\tau} = -I^- + S^- + \frac{a}{2}\int_0^1 (1+3g\bar{u}u')\cdot I^- \cdot du' + \frac{a}{2}\int_{-1}^0 (1+3g\bar{u}u')\cdot I^+ \cdot du'$$

$$-\bar{u}\frac{dI^+}{d\tau} = -I^+ + S^+ + \frac{a}{2}\int_0^1 (1+3g\bar{u}u')\cdot I^- \cdot du' + \frac{a}{2}\int_{-1}^0 (1+3g\bar{u}u')\cdot I^+ \cdot du'$$

No single answer for best value—the 2-stream approximation has different “flavors”.

In general, $0.5 \le \bar{u} \le 0.7$

Two definitions:

$$\bar{u} = \frac{\int_0^1 uI \cdot du}{\int_0^1 I \cdot du} = 0.5 \qquad \bar{u} = \sqrt{\langle u^2 \rangle} = \sqrt{\frac{\int_0^1 u^2 I \cdot du}{\int_0^1 I \cdot du}} \approx \frac{1}{\sqrt{3}} = 0.577$$

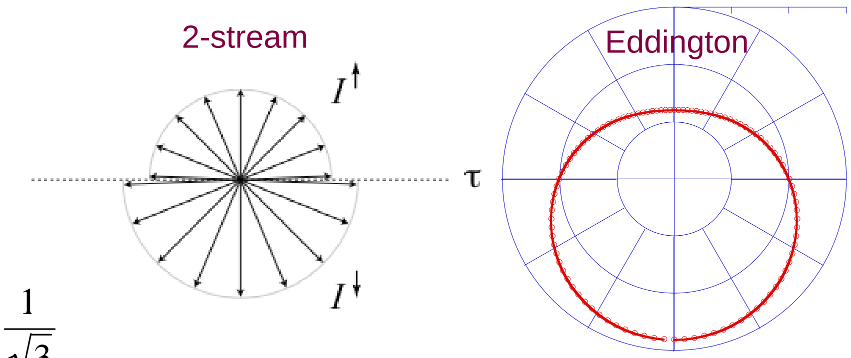
Some flavors are better in one case, others in other cases. In general, a comparison study found that 0.577 is the best overall choice.

Eddington approximation

Approximations (1)-(3) are the same as for 2-stream (no emission, azimuthal averaging, 2-term Legendre phase fn)
Only difference is in approximation (4).

Approx 4: *I* is separated into two components differently than in 2-stream: Instead of I^+ and I^-, we have I^0 and I^1 so that

$$I(\tau,u) = I_0(\tau) + uI_1(\tau)$$



Approx.5 is same as in one flavor of 2-stream: $\bar{u} = \frac{1}{\sqrt{3}}$

Still, same basic procedure applies, and at the end, we get to a diffusion equation in an identical shape as for 2-stream approximation. In fact, one can show that Eddington approximation equations are equivalent to the 2-stream equations if that one uses $\bar{u} = \frac{1}{\sqrt{3}}$

as if Eddington was a special "flavor" of 2-stream (only the coefficient values are different, since we have I^0 and I^1 instead of I^+ and I^-).

More Streams Methods

Eddington can be expanded into 4 or 6 streams much like 2-stream. For 4-stream, Li J, Ramaswamy V: Four-stream spherical harmonic expansion approximation for solar radiative transfer, *J. Atmos. Sci*., **53**, 1174-1186, 1996)

The word “two-stream” refers to two-point quadrature, that is, *I*-values

$$\mu \frac{dI}{d\tau} = -I + (1-a)B + \frac{a}{4\pi} P(\Omega_0,\Omega)\cdot F^s \cdot e^{-\frac{\tau}{\mu_0}} + \frac{a}{4\pi}\int_{4\pi} P(\Omega',\Omega)\cdot I(\Omega')\cdot d\Omega$$

$$\bar{u}\frac{dI^-}{d\tau} = -I^- + S^- + \frac{a}{2}I^- b + \frac{a}{2}I^+(1-b)$$

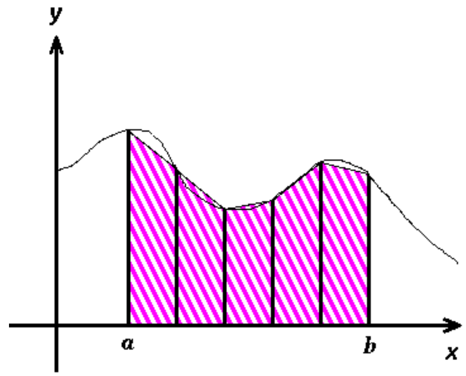
Improvements to basic 2-stream and Eddington methods:

- Accuracy (comparisons to other methods): e.g., more terms, complex phase functions
- 3D (e.g., multiple layers, cuboidal clouds, spherical shell atmosphere)
- Combine with single scattering approximation for azimuthal dependence

Discrete Ordinates Method: an Extension of Two-Stream

An example for quadrature formulas (which approximate integrals by finite sums): $\int^{1} I(u)\cdot du = \sum^{N} w_j \cdot I_j$

One quadrature is trapezoidal rule: $\int_{-1}^{1} I(u)\cdot du \approx \Delta u\left(\frac{1}{2}I_1 + I_2 + I_3 + \ldots + \frac{1}{2}I_N\right)$



In practice, double Gaussian quadrature is more practical

Where did we use such quadrature formulas?—2-stream

2-stream: $I^+, I^- \Rightarrow I(\bar{u}),\ I(-\bar{u})$

The integral in $\mu\frac{dI}{d\tau} = -I + (1-a)B + \frac{a}{4\pi}P\cdot F^s \cdot e^{-\frac{\tau}{\mu_0}} + \frac{a}{4\pi}\int_{4\pi} P\cdot I\cdot d\Omega$

is approximated as:

$\bar{u}\frac{dI^-}{d\tau} = -I^- + S^- + \frac{a}{2}I^- b + \frac{a}{2}I^+(1-b)$, where *b* is forward scattering coeff. (see previous lecture)

Discrete Ordinates: $u_i\frac{dI_i^-}{d\tau} = -I_i^- + S^- + \frac{a}{2}\sum_{j=1}^{N}\alpha_j I_j$ *i*=1,2,…,*N* (*N* typically > 48)

Boundary conditions:

Set of coupled equations, solved by iterations. (Intermediate methods: 4-stream, 6-stream)

Good public code (DISORT) available, and so very widely used (Tamás)
Good online calculator (SBDART) available at http://arm.mrcsb.com/sbdart/

Application in GCMs

Because of their speed, these two approximations dominate in GCM solar radiation calculations.

Even using these fast methods, radiative calculations are often not performed at every time step or at every grid point.

Despite the speed of these methods, radiative calculations still take up huge chunks of the climate simulations (e.g., a third or a half).

$$I^-(\tau) = X_1 e^{k\tau} + Y_1 e^{-k\tau} + W_1 e^{-\frac{\tau}{\mu_0}} \Rightarrow F^- =$$

4-step procedure of GCM calculations:

$$I^+(\tau) = X_2 e^{k\tau} + Y_2 e^{-k\tau} + W_2 e^{-\frac{\tau}{\mu_0}} \qquad F^+ =$$

Step 1: Find the single scattering properties of each model layer by combining the effects of each constituent in that layer:

$$\tau = \sum_i \tau_i \qquad a = \frac{\sum_i a_i \tau_i}{\tau} \qquad g = \frac{\sum_i g_i a_i \tau_i}{a\tau} \qquad f = \frac{\sum_i f_i a_i \tau_i}{a\tau}$$

Step 2: Use δ-2 stream or δ-Eddington equations to calculate radiative properties of each layer.

Step 3: Link the layers radiatively to get fluxes at each level in the model.

Step 4: Repeat Steps 1-3 for each wavelength. Obtain broadband results by combining results for each wavelength. Convert fluxes into heating rates for each layer.

Single scattering approximation

Useful in what situations?

Key approximation: $S(\tau)=Q(\tau)=(1-a)B+\frac{a}{4\pi}F^{s}\cdot e^{-\frac{\tau}{u_0}}$

If we include phase function: $S(\tau,\mu,\mu_0,\varphi-\varphi_0)=Q(\tau,\mu,\mu_0,\varphi-\varphi_0)=(1-a)B+\frac{a}{4\pi}F^{s}\cdot e^{-\frac{\tau}{u_0}}\cdot P(\mu,\mu_0,\varphi-\varphi_0)$

Method: first we calculate *S*, then we can calculate any radiances by substituting the results into the equations

$$I^{-}(\tau,u)=\frac{1}{u}\int_0^{\tau}S(\tau')\cdot e^{-\frac{\tau'-\tau}{u}}\cdot d\tau' \qquad I^{+}(\tau,u)=\frac{1}{u}\int_{\tau}^{\tau^*}S(\tau')\cdot e^{-\frac{\tau-\tau'}{|u|}}\cdot d\tau'$$

Carrying out the integration gives:
(assuming vertically constant *P*)

$$I_d^{-}(\tau,u,\varphi)=(1-a)B\left(1-e^{-\frac{\tau}{u}}\right)+\frac{au_0}{4\pi(u_0-u)}F^{s}\cdot P(u,u_0,\varphi-\varphi_0)\cdot\left[e^{-\frac{\tau}{u_0}}-e^{-\frac{\tau}{u}}\right]$$

$$I^{+}(\tau,u,\varphi)=(1-a)B\left(1-e^{\frac{(\tau^*-\tau)}{u}}\right)+\frac{au_0}{4\pi(u_0-u)}F^{s}\cdot P(u,u_0,\varphi-\varphi_0)\cdot\left[e^{-\frac{\tau}{u_0}}-e^{-\left(\frac{\tau^*-\tau}{u}+\frac{\tau^*}{u_0}\right)}\right]$$

We assume vertically uniform *B*-value

Advantages:
•Fast and simple
•Works well for any phase function

Successive orders of scattering method

Iterative method: start with single scattering approximation, then move to higher and higher orders of scattering.

Get $S(\tau)$ as: $S(\tau) = S_1(\tau) + S_2(\tau) + S_3(\tau) + \ldots + S_N(\tau)$

Let's keep in mind the integral form of RTE: $S(\tau) = Q(\tau) + \frac{a}{2}\int_0^{\tau^*} S(\tau') \cdot E_1(|\tau - \tau'|) \cdot d\tau'$

Step 1. (= single scattering approximation): $S_1(\tau) = Q(\tau) = (1-a)B + \frac{a}{4\pi} F^s \cdot e^{-\frac{\tau}{u_0}}$

Step 2: $S_2(\tau) = S_1(\tau) + \frac{a}{2}\int_0^{\tau^*} S_1(\tau') \cdot E_1(|\tau - \tau'|) \cdot d\tau' \left(= Q(\tau) + \frac{a}{2}\int_0^{\tau^*} Q(\tau') \cdot E_1(|\tau - \tau'|) \cdot d\tau' \right)$

Step 3:
$$\begin{cases} S_3(\tau) = S_2(\tau) + \frac{a}{2}\int_0^{\tau^*} S_2(\tau') \cdot E_1(|\tau - \tau'|) \cdot d\tau' = \\ = S_1(\tau) + \frac{a}{2}\int_0^{\tau^*} S_1(\tau') \cdot E_1(|\tau - \tau'|) \cdot d\tau' + \left(\frac{a}{2}\right)^2 \int_0^{\tau^*} E_1(|\tau - \tau_1'|) \int_0^{\tau^*} E_1(|\tau_1 - \tau_2|) \cdot S_1(\tau_2) \cdot d\tau_2 \cdot d\tau_1 \end{cases}$$

Step N: $S_N(\tau) = S_{N-1}(\tau) + \frac{a}{2}\int_0^{\tau^*} S_{N-1}(\tau') \cdot E_1(|\tau - \tau'|) \cdot d\tau'$

If we substitute S_{N-1}, then S_{N-2},… so that only S_1 will remain: S is expressed as a Neumann series expansion of $Q(\tau)$.

Successive orders of scattering method, part 2

Method has the same advantages as single scattering approximation, but can also account for higher-order scattering.
Disadvantage: computational time (this is why it usually is an approximation)

Sum of Neumann series can be approximated in certain cases, e.g., “cooling to space approximation” (IR cooling to space, no downwelling IR):

$$S(\tau,\mu,\mu_0,\varphi-\varphi_0) \approx (1-a)B + \frac{\frac{a}{4\pi} F^s \cdot P(\mu,\mu_0,\varphi-\varphi_0) \cdot e^{-\frac{\tau}{u_0}}}{1-a\left[1-\frac{E_2(\tau^*/2)}{2}\right]}$$

where $$E_n(x) = \int_0^1 u^{n-2} \cdot e^{-\frac{x}{u}} \cdot du$$

Adding-doubling method

Adding-doubling is used in some dynamical models, creating satellite retrieval tables.

We start with a thin layer (e.g., $\tau = 10^{-8}$), so that we can ignore last term in RTE:

$$\mu \frac{dI}{d\tau} = -I + (1-a)B + \frac{a}{4\pi} P(\Omega_0, \Omega) \cdot F^s \cdot e^{-\frac{\tau}{\mu_0}} + \cancel{\frac{a}{4\pi} \int_{4\pi} P(\Omega', \Omega) \cdot I(\Omega') \cdot d\Omega}$$

The remaining equation is easy to solve for the thin layer.

From thin layer, build thick layer by developing rules on how two layers can be combined:
ρ_1, ρ_2, T_1, T_2 => ρ_{1+2}, T_{1+2}

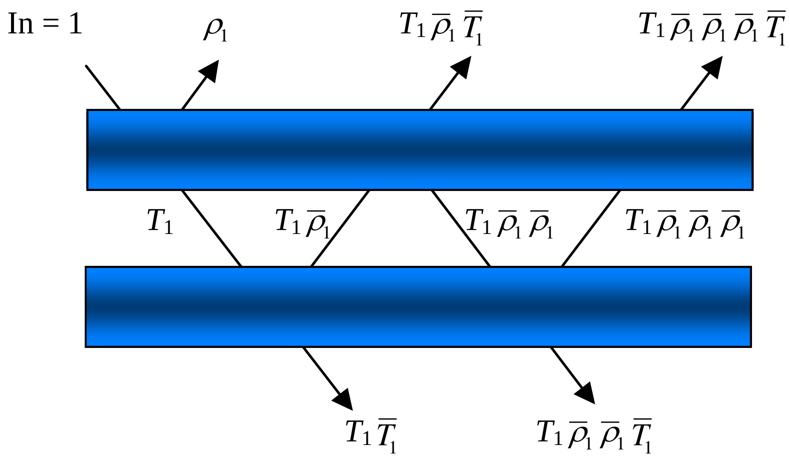
Once two layers are combined into one, it again can be combined with another one, until the desired thickness is achieved.

Doubling: if the two layers we combine are identical
Adding: if the two layers have different properties
Either way, we use same basic approach

For simplicity, we discuss the method of combining layers only for isotropic scattering.
In isotropic case, radiance and flux are easy to relate:

Doubling

Known parameters of a single layer: T_1, $\overline{T}_1$, ρ_1 and $\overline{\rho}_1$



$$\rho_t = \rho_1 + T_1\overline{\rho}_1\overline{T}_1 + T_1\overline{\rho}_1\overline{\rho}_1\overline{\rho}_1\overline{T}_1 + T_1\overline{\rho}_1\overline{\rho}_1\overline{\rho}_1\overline{\rho}_1\overline{\rho}_1\overline{T}_1 + \ldots =$$
$$= \rho_1 + T_1\overline{\rho}_1\left(1 + \overline{\rho}_1\overline{\rho}_1 + \overline{\rho}_1\overline{\rho}_1\overline{\rho}_1\overline{\rho}_1 + \ldots\right)\overline{T}_1 =$$
$$= \rho_1 + T_1\overline{\rho}_1\frac{1}{\left(1-\overline{\rho}_1^{\,2}\right)}\overline{T}_1$$

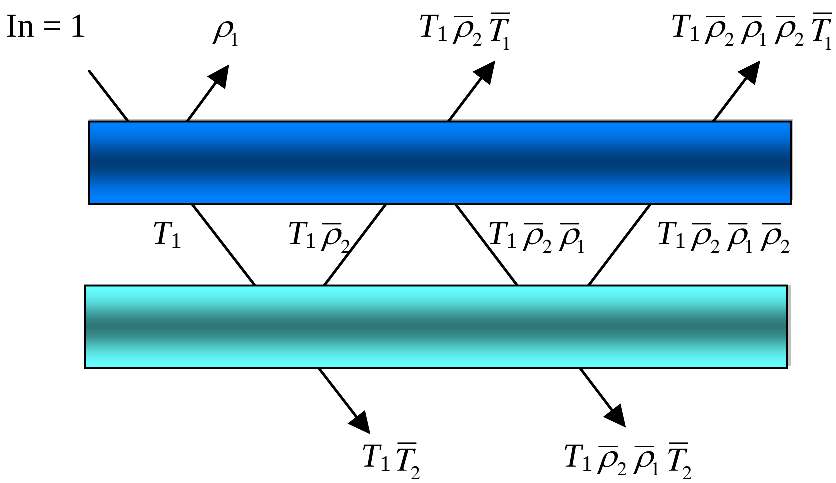
Here, we used that the sum of a geometric series (for any 0<x<1) is: $x^0 + x^1 + x^2 + x^3 + \ldots = \dfrac{1}{1-x}$
(in our case, $x = \overline{\rho}_1\overline{\rho}_1$)

Similarly, the total transmittance is?:

$$T_t = T_1\overline{T}_1 + T_1\overline{\rho}_1\overline{\rho}_1\overline{T}_1 + T_1\overline{\rho}_1\overline{\rho}_1\overline{\rho}_1\overline{\rho}_1\overline{T}_1 + \ldots =$$
$$= T_1\left(1 + \overline{\rho}_1\overline{\rho}_1 + \overline{\rho}_1\overline{\rho}_1\overline{\rho}_1\overline{\rho}_1 + \ldots\right)\overline{T}_1 =$$
$$= T_1\frac{1}{\left(1-\overline{\rho}_1^{\,2}\right)}\overline{T}_1$$

Adding

Difference from doubling: two layers have different properties



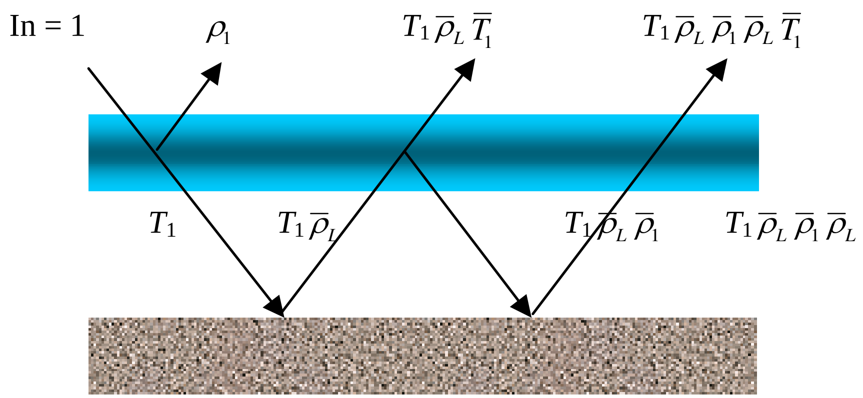
$$\rho_t = \rho_1 + T_1\bar{\rho}_2\bar{T}_1 + T_1\bar{\rho}_2\bar{\rho}_1\bar{\rho}_2\bar{T}_1 + T_1\bar{\rho}_2\bar{\rho}_1\bar{\rho}_2\bar{\rho}_1\bar{\rho}_2\bar{T}_1 + \ldots =$$
$$= \rho_1 + T_1\bar{\rho}_2\left(1 + \bar{\rho}_1\bar{\rho}_2 + \bar{\rho}_1\bar{\rho}_2\bar{\rho}_1\bar{\rho}_2 + \ldots\right)\bar{T}_1 =$$
$$= \rho_1 + T_1\bar{\rho}_2\frac{1}{\left(1 - \bar{\rho}_1\bar{\rho}_2\right)}\bar{T}_1$$

$$T_t = T_1\bar{T}_2 + T_1\bar{\rho}_2\bar{\rho}_1\bar{T}_2 + T_1\bar{\rho}_2\bar{\rho}_1\bar{\rho}_2\bar{\rho}_1\bar{T}_2 + \ldots =$$
$$= T_1\left(1 + \bar{\rho}_2\bar{\rho}_1 + \bar{\rho}_2\bar{\rho}_1\bar{\rho}_2\bar{\rho}_1 + \ldots\right)\bar{T}_2 =$$
$$= T_1\frac{1}{\left(1 - \bar{\rho}_2\bar{\rho}_1\right)}\bar{T}_2$$

Adding surface

For example, for including Lambertian surface reflection into 2-stream or Eddington calculations

Surface albedo: $\bar{\rho}_L$



$$
\begin{aligned}
\rho_t &= \rho_1 + T_1\bar{\rho}_L\bar{T}_1 + T_1\bar{\rho}_L\bar{\rho}_1\bar{\rho}_L\bar{T}_1 + T_1\bar{\rho}_L\bar{\rho}_1\bar{\rho}_L\bar{\rho}_1\bar{\rho}_L\bar{T}_1 + \ldots = \\
&= \rho_1 + T_1\bar{\rho}_L\left(1 + \bar{\rho}_1\bar{\rho}_L + \bar{\rho}_1\bar{\rho}_L\bar{\rho}_1\bar{\rho}_L + \ldots\right)\bar{T}_1 = \\
&= \rho_1 + T_1\bar{\rho}_L \frac{1}{\left(1 - \bar{\rho}_1\bar{\rho}_L\right)}\bar{T}_1
\end{aligned}
$$

$$
\begin{aligned}
T_t &= T_1 + T_1\bar{\rho}_L\bar{\rho}_1 + T_1\bar{\rho}_L\bar{\rho}_1\bar{\rho}_L\bar{\rho}_1 + \ldots = \\
&= T_1\left(1 + \bar{\rho}_L\bar{\rho}_1 + \bar{\rho}_L\bar{\rho}_1\bar{\rho}_L\bar{\rho}_1 + \ldots\right) = \\
&= T_1\frac{1}{\left(1 - \bar{\rho}_L\bar{\rho}_1\right)}
\end{aligned}
$$

Adding-Doubling Techniques

- The advantage of the technique is that one can start with thin layers – small optical depths – which are easier to handle mathematically. There is no limit as to how many of the layers one eventually adds.

- If the two layers are dissimilar, then we must take into account that the transmittance and reflectance will be different for illumination from above and below.

Limitations of accurate methods discussed so far

(Adding-Doubling, Discrete Ordinates, Spherical harmonics)

•1D medium

•Require fairly simple phase functions (e.g., spherical particles)

•Do not give information of photon paths

•No time-dependence of photon journey (e.g., for pulse-stretching of lidar signal)

(though it can be deduced by including various levels of gas absorption)