Figure 5.9 A beam of radiation is incident on an absorbing/emitting region at the boundary point $P_1$. It is attenuated along the path $P_1P_2$ and emerges at the point $P_2$. The propagation direction of the beam is denoted by $\hat{\Omega}$. In addition, thermal emission adds to the beam at all points within the medium.
Collimated Incidence - Lambert Surface

- If the incident light is direct sunlight then

\[
I^{-}(\Omega) = F^{S} \delta(\Omega - \Omega_{0}) = F^{S} (\cos \theta - \cos \theta_{0}) \delta(\phi - \phi_{0})
\]

The incident flux is given by \( F^{-} = F^{S} \cos \theta_{0} \)

Hence \( I_{r}^{+} = \rho_{L} \cos \theta_{0} F^{S} \)

For a collimated beam the intensity reflected from a Lambert surface is proportional to the cosine of the angle of incidence.
Collimated Incidence – Specular reflectance

- Here the reflected intensity is directed along the angle of reflection only.
- Hence $\theta' = \theta$ and $\phi = \phi' + \pi$
- Spectral reflection function $\rho_S(v, \theta)$

\[
I_r^+(\hat{\Omega}) = \rho_S(\theta) F^S \delta(\cos \theta_0 - \cos \theta) \delta(\phi - [\phi_0 + \pi])
\]

- and the reflected flux:

\[
F_{r}^+ = \rho_S(\theta_0) F^S \cos \theta_0
\]
Differential equation of Radiative Transfer

- Consider conservative scattering & no change in frequency.
- Assume the incident radiation is collimated.
- We now need to look more closely at the secondary ‘emission’ that results from scattering. Remember that from the definition of the intensity that

\[ d^4E^+ = I_\nu(\hat{\Omega}')dAdtd\nu d\omega \]
Differential Equation of Radiative Transfer

- The radiative energy scattered in all directions is
\[ \sigma \, ds \, d^4 E' \]

- We are interested in that fraction of the scattered energy that is directed into the solid angle \( d\omega \) centered about the direction \( \Omega \).

- This fraction is proportional to
\[ p(\hat{\Omega}', \hat{\Omega}) \, d\omega / 4\pi \]
Differential Equation of Radiative Transfer

• If we multiply the scattered energy by this fraction and then integrate over all incoming angles, we get the total scattered energy emerging from the volume element in the direction $\mathbf{\Omega}$,

$$d^4 E = \sigma(\nu) dV \, dt \, d\nu \, d\omega \int d\omega' \frac{p(\hat{\mathbf{\Omega}'},\hat{\mathbf{\Omega}})}{4\pi} I_{\nu}(\hat{\mathbf{\Omega}}')$$

• The emission coefficient for scattering is

$$j^SC_{\nu} = \frac{d^4 E}{dV \, dt \, d\nu \, d\omega} = \sigma(\nu) \int d\omega' \frac{p(\hat{\mathbf{\Omega}'},\hat{\mathbf{\Omega}})}{4\pi} I_{\nu}(\hat{\mathbf{\Omega}}')$$
Differential Equation of Radiative Transfer

• The source function for scattering is thus

\[ S_{v}^{SC} (\hat{r},\hat{\Omega}) = \frac{j_{v}^{SC}}{k(v)} = \frac{\sigma(v)}{k(v)} \int \frac{d\omega'}{4\pi} p(\hat{\Omega}',\hat{\Omega}) I_{v}(\hat{\Omega}') \]

• The quantity \( \sigma(v)/k(v) \) is called the single-scattering albedo and given the symbol \( a(v) \).

• If thermal emission is involved, \( (1-a) \) is the volume emittance \( \varepsilon \).
Differential Equation of Radiative Transfer

• The complete time-independent radiative transfer equation which includes both multiple scattering and absorption is

\[
\frac{dI_v}{d\tau_s} = I_v + [1 - a(\nu)]B_v(T) + \frac{a(\nu)}{4\pi} \int d\omega' p(\hat{\Omega}', \hat{\Omega}) I_v
\]
Solution for Zero Scattering

- If there is no scattering, e.g. in the thermal infrared, then the equation becomes

\[ \frac{dI_v}{d\tau_S} = -I_v + B_v(T) \]

- This equation can be easily integrated using an integrating factor \( e^\tau \)

\[ \frac{dI}{d\tau}e^\tau + Ie^\tau = \frac{d}{d\tau}(Ie^\tau) = Be^\tau \]
Solution for Zero Scattering

- Consider a straight path between point $P_1$ and $P_2$. The optical path from $P_1$ to an intermediate point $P$ is given by

\[
\tau(P_1, P) = \int_{P_1}^{P} \alpha \, ds = \int_{0}^{P} \alpha \, ds - \int_{0}^{P_1} \alpha \, ds = \tau(P) - \tau(P_1)
\]

- Integrating along the path from $P_1$ to $P_2$

\[
\int_{\tau(P_1)}^{\tau(P_2)} \frac{d}{dt}(Ie^{\tau}) = I[\tau(P_2)]e^{\tau(P_2)} - I[\tau(P_1)]e^{\tau(P_1)}
\]

\[
= \int_{\tau(P_1)}^{\tau(P_2)} d\tau B(\tau)e^{\tau}
\]
Solution for Zero Scattering

• Dividing through by $e^{\tau(P_2)}$ we get

\[
I[\tau(P_2)] = I[\tau(P_1)]e^{-\tau(P_2)+\tau(P_1)} + \int \frac{\tau(P_2)}{\tau(P_1)} dt B(t)e^{\tau-P_2}
\]

\[
= I[\tau(P_1)]e^{-\tau(P_1,P_2)} + \int d\tau B(\tau)e^{-\tau(P,P_2)}
\]

• But what does this equation tell us about the physics of the problem?
Physical Description

\[ I[\tau(P_1)] \]

\[ \tau(P_1) \]

\[ \tau \exp(-[\tau(P_2) - \tau(P_1)]) \]

\[ B(\tau)\Delta \tau \exp(-[\tau(P_2) - \tau]) \]
Solution for Zero Scattering

• Break up the path from $P_1$ to $P_2$ into small elements $\delta s$ with optical depths $\delta \tau$
• When $\sigma_n$ is zero then $\varepsilon_n$ is equal to 1
• Hence $\delta \tau B$ is the blackbody emission from the element $ds$
• The intensity at $P_1$ is $I[\tau(P_1)]$
• This intensity will be absorbed as it moves from $P_1$ to $P_2$, and the intensity at $P_2$ will be $I[\tau(P_1)] \exp[-(\tau(P_2) - \tau(P_1))]$
Solution for Zero Scattering

• Now consider each small element $P$ with a $\Delta \tau$, with an optical depth $\tau$
• Emission from each element is $B \Delta \tau$
• The amount of this radiation that reaches $P_2$ is $B \Delta \tau \exp[-\tau(P,P_2)]$ where $\tau$ is the optical depth between $P$ and $P_2$
• Hence the total amount of radiation reaching $P_2$ from all elements is

$$\sum_{0}^{n} \Delta \tau B(\tau)e^{-\tau(P,P_2)} \equiv \int_{\tau(P_1)}^{\tau(P_2)} B(\tau)e^{-\tau(P,P_2)} \delta\tau$$
Isothermal Medium – Arbitrary Geometry

Redefine the origin such that \( \tau(P_1) = 0 \), then

\[
I[\tau(P_2)] = I[\tau(P_1)]e^{-\tau(P_2)} + B \int_0^{\tau(P_2)} dt e^{-(\tau(P_2) - \tau)}
\]

\[
= I[0]e^{-\tau(P_2)} + B[1 - e^{-\tau(P_2)}]
\]

If the medium is optically thin, i.e. \( \tau(P_2) \ll 1 \) then the second term becomes \( B \tau(P_2) \).

If there is no absorption or scattering then \( \tau=0 \) and the intensity in any direction is a constant, i.e. \( I[\tau(P_2)] = I[\tau(P_1)] \).
Isothermal Medium – Arbitrary Geometry

If we consider the case when $\tau >> 1$ then the total intensity is equal to $B(T)$. In this case the medium acts like a blackbody in all frequencies, i.e. is in a state of thermodynamic equilibrium.

If ones look toward the horizon then in a homogeneous atmosphere the atmosphere has a constant temperature. Hence the observed intensity is also blackbody
Zero Scattering in Slab Geometry

• Most common geometry in the theory of radiative transfer is a plane-parallel medium or a slab.
• The vertical optical path (optical depth) is given the symbol $\tau$ as distinct from the slant optical path $\tau_s$.
• Using $z$ as altitude $\tau(z) = \tau_s |\cos\theta| = \tau_s \mu$.
• The optical depth is measured along the vertical downward direction, i.e. from the ‘top’ of the medium.
Half-range Intensities

Figure 5.10 Half-range intensities in a slab geometry. The optical depth variable $\tau$ is measured downward from the “top” of the medium ($\tau = 0$) to the “bottom” ($\tau = \tau^*$). $\mu$ is equal to the absolute value of the cosine of the angle $\theta$, the polar angle of the propagation vector $\hat{\Omega}$. 
Half-Range Quantities in Slab Geometry

- The *half-range intensities* are defined by:

\[
I^+_{v}(\tau, \theta, \phi) \equiv I_{v}(\tau, \theta \leq \pi/2, \phi) \\
I^-_{v}(\tau, \theta, \phi) \equiv I_{v}(\tau, \theta > \pi/2, \phi)
\]

- Note that the negative direction is for the downward flux,
Half-Range Quantities in Slab Geometry

- The radiative flux is also defined in terms of half-range quantities.

\[ F_v^+ = \int d\omega \cos \theta I_v^+(\hat{\Omega}) = \int_0^{2\pi} d\phi \int_0^{\pi/2} d\theta \sin \theta \cos \theta I_v^+(\tau, \theta, \phi) \]
\[ = \int_0^{2\pi} d\phi \int_0^1 d\mu \mu I_v^+(\tau, \mu, \phi) \]

\[ F_v^- = -\int d\omega \cos \theta I_v^-(\hat{\Omega}) = -\int_0^{2\pi} d\phi \int_0^{\pi/2} d\theta \sin \theta \cos \theta I_v^-(\tau, \theta, \phi) \]
\[ = \int_0^{2\pi} d\phi \int_0^1 d\mu \mu I_v^-(\tau, \mu, \phi) \]
Half Range Quantities

• In the limit of no scattering the radiative transfer equations for the half-range intensities become

\[
\mu \frac{dI^+_{\nu}(\tau, \mu, \phi)}{d\tau} = I^+_{\nu}(\tau, \mu, \phi) - B(\tau)
\]

\[
-\mu \frac{dI^-_{\nu}(\tau, \mu, \phi)}{d\tau} = I^-_{\nu}(\tau, \mu, \phi) - B(\tau)
\]
Formal Solution in Slab Geometry

- Choose the integrating factor $e^{\tau/\mu}$, for the first equation, then

$$\frac{d}{d\tau}(I_v e^{\tau/\mu}) = \left(\frac{dI_v}{d\tau} + \frac{1}{\mu} I_v\right) e^{\tau/\mu} = \frac{B_v(\tau)}{\mu} e^{\tau/\mu}$$

- This represents a downward beam so we integrate from the “top” of the atmosphere ($\tau=0$) to the bottom ($\tau=\tau^*$).
Slab geometry

\[
\int_{0}^{\tau^*} \frac{d}{d\tau'} \left( I_v^- e^{\tau'/\mu} \right) = I_v^- (\tau^*, \mu, \phi) e^{\tau^*/\mu} - I_v^- (0, \mu, \phi)
\]

or

\[
I_v^- (\tau^*, \mu, \phi) = I_v^- (0, \mu, \phi) e^{-\tau^*/\mu} + \int_{0}^{\tau^*} \frac{d\tau'}{\mu} B_v(\tau') e^{-(\tau^* - \tau')/\mu}
\]
Slab Geometry

- For an interior point, $\tau < \tau^*$, we integrate from 0 to $\tau$. The solution is easily found by replacing $\tau^*$ with $\tau$

$$I_v(\tau, \mu, \phi) = I_v(0, \mu, \phi)e^{-\tau/\mu} + \int_0^\tau \frac{d\tau'}{\mu} B_v(\tau')e^{-(\tau-\tau')/\mu}$$
Mid-Term Exam
March 15