AOSC621
RTE for None-absorbing Medium

Lesson 8

Solution for Zero Scattering

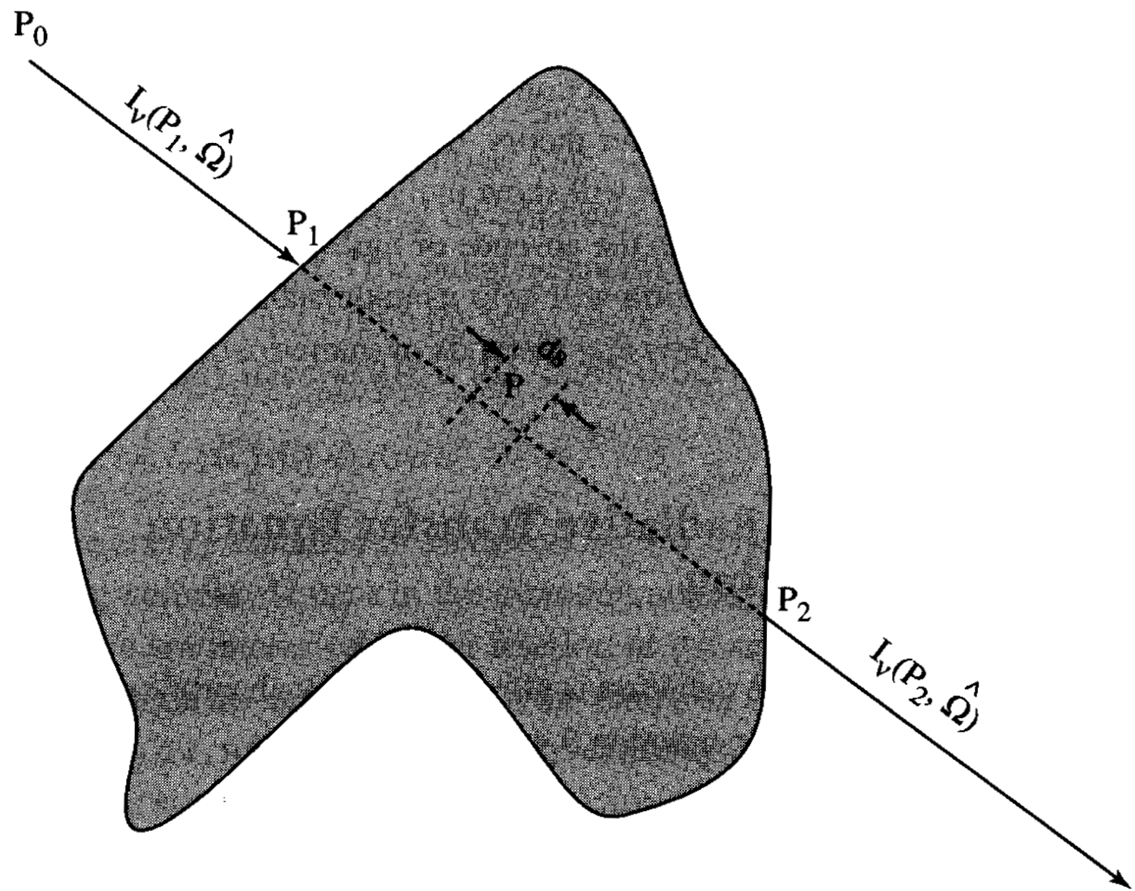


Figure 5.9 A beam of radiation is incident on an absorbing/emitting region at the boundary point P_1. It is attenuated along the path $P_1 P_2$ and emerges at the point P_2. The propagation direction of the beam is denoted by $\hat{\Omega}$. In addition, thermal emission adds to the beam at all points within the medium.

Collimated Incidence - Lambert Surface

- If the incident light is direct sunlight then

$$I^{-}(\widehat{\Omega}) = F^{S}\delta(\widehat{\Omega} - \widehat{\Omega}_0) = F^{S}(\cos\theta - \cos\theta_0)\delta(\phi - \varphi_0)$$

The incident flux is given by $F^{-} = F^{S}\cos\theta_0$

Hence $\quad I_r^{+} = \rho_L \cos\theta_0 F^{S}$

For a collimated beam the intensity reflected from a Lambert surface is proportional to the cosine of the angle of incidence.

Collimated Incidence – Specular reflectance

- Here the reflected intensity is directed along the angle of reflection only.
- Hence $\theta' = \theta$ and $\phi = \phi' + \pi$
- Spectral reflection function $\rho_S(\nu, \theta)$

$$I_r^+(\hat{\Omega}) = \rho_S(\theta) F^S \delta(\cos\theta_0 - \cos\theta)\delta(\phi - [\phi_0 + \pi])$$

- and the reflected flux:

$$F_r^+ = \rho_S(\theta_0) F^S \cos\theta_0$$

Differential equation of Radiative Transfer

- Consider conservative scattering & no change in frequency.
- Assume the incident radiation is collimated
- We now need to look more closely at the secondary ‘emission’ that results from scattering. Remember that from the definition of the intensity that

$$d^4E^+ = I_\nu(\hat{\Omega}')\,dA\,dt\,d\nu\,d\omega$$

Differential Equation of Radiative Transfer

- The radiative energy scattered in *all* directions is

$$\sigma \, \mathrm{ds} \; d^4 E'$$

- We are interested in that fraction of the scattered energy that is directed into the solid angle $d\omega$ centered about the direction Ω.

- This fraction is proportional to

$$p(\hat{\Omega}', \hat{\Omega}) \, d\omega / 4\pi$$

Differential Equation of Radiative Transfer

- If we multiply the scattered energy by this fraction and then integrate over all incoming angles, we get the total scattered energy emerging from the volume element in the direction *Ω*,

$$d^4 E = \sigma(\nu)\, dV\, dt\, d\nu\, d\omega \int_{4\pi} d\omega' \frac{p(\hat{\Omega}', \hat{\Omega})}{4\pi} I_\nu'(\hat{\Omega}')$$

- The emission coefficient for scattering is

$$j_\nu^{SC} \equiv \frac{d^4 E}{dV\, dt\, d\nu\, d\omega} = \sigma(\nu) \int_{4\pi} \frac{d\omega'}{4\pi} p(\hat{\Omega}', \hat{\Omega}) I_\nu(\hat{\Omega}')$$

Differential Equation of Radiative Transfer

- The source function for scattering is thus

$$S_\nu^{SC}(\hat{r},\hat{\Omega}) = \frac{j_\nu^{SC}}{k(\nu)} = \frac{\sigma(\nu)}{k(\nu)} \int_{4\pi} \frac{d\omega'}{4\pi} p(\hat{\Omega}',\hat{\Omega}) I_\nu(\hat{\Omega}')$$

- The quantity *σ(ν)/k(ν)* is called the single-scattering albedo and given the symbol *a(ν).*

- If thermal emission is involved, (1-*a*) is the volume emittance ε.

Differential Equation of Radiative Transfer

- The complete time-independent radiative transfer equation which includes both multiple scattering and absorption is

$$\frac{dI_\nu}{d\tau_s} = -I_\nu + \left[1 - a(\nu)\right] B_\nu(T) + \frac{a(\nu)}{4\pi} \int_{4\pi} d\omega' p(\hat{\Omega}', \hat{\Omega}) I_\nu$$

Solution for Zero Scattering

- If there is no scattering, e.g. in the thermal infrared, then the equation becomes

$$\frac{dI_\nu}{d\tau_S} = -I_\nu + B_\nu(T)$$

- This equation can be easily integrated using an integrating factor e^τ

$$\frac{dI}{d\tau}e^\tau + Ie^\tau = \frac{d}{d\tau}(Ie^\tau) = Be^\tau$$

Solution for Zero Scattering

- Consider a straight path between point P_1 and P_2 . The optical path from P_1 to an intermediate point P is given by

$$\tau(P_1,P)=\int_{P_1}^{P}\alpha\,ds=\int_{0}^{P}\alpha\,ds-\int_{0}^{P_1}\alpha\,ds\equiv\tau(P)-\tau(P_1)$$

- Integrating along the path from P_1 *to* P_2

$$\int_{\tau(P_1)}^{\tau(P_2)}dt\,\frac{d}{dt}(Ie^{\tau})=I\left[\tau(P_2)\right]e^{\tau(P_2)}-I\left[\tau(P_1)\right]e^{\tau(P_1)}$$

$$=\int_{\tau(P_1)}^{\tau(P_2)}d\tau B(\tau)e^{\tau}$$

Solution for Zero Scattering

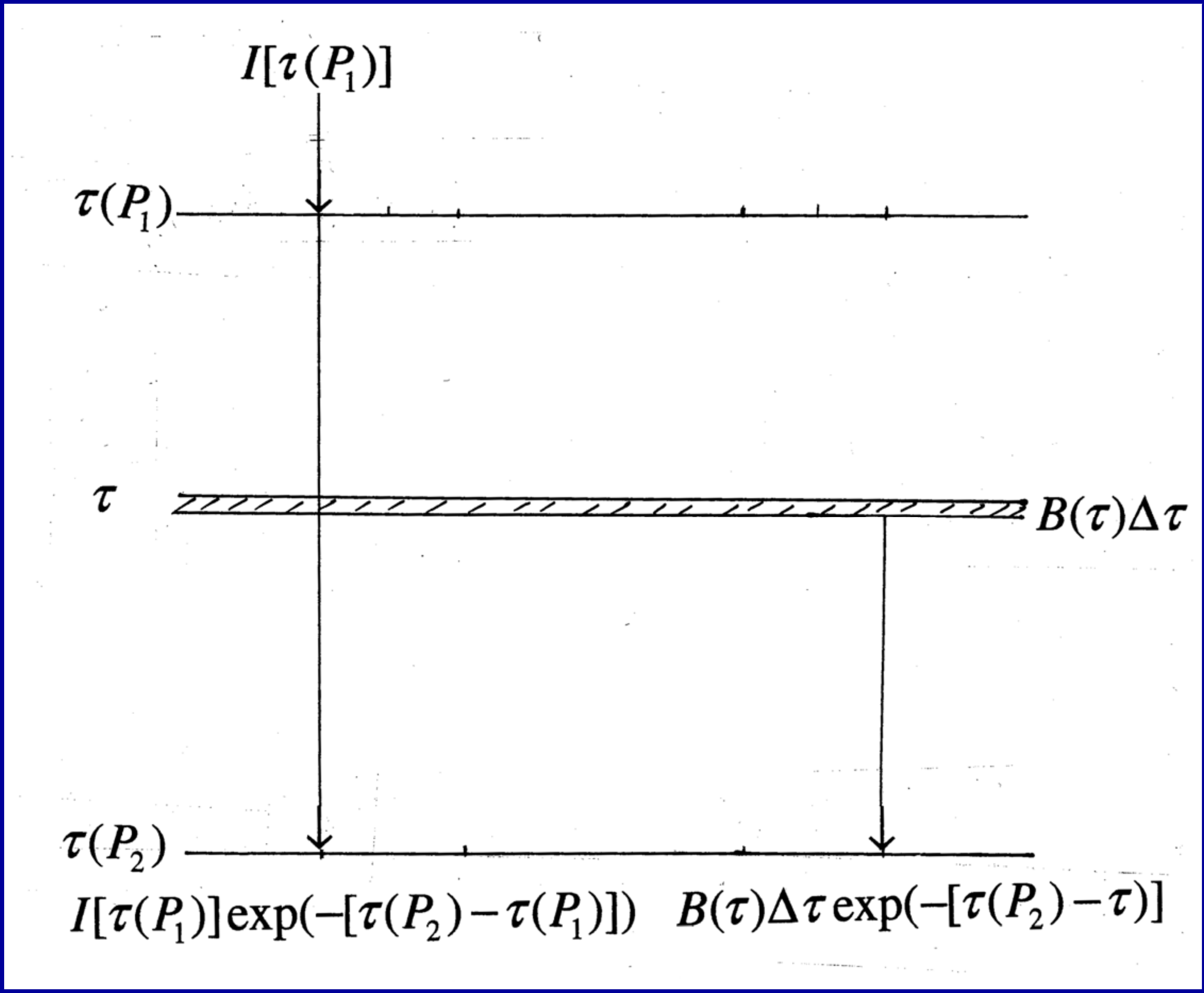
- Dividing through by $e^{\tau(P2)}$ we get

$$I\left[\tau(P_2)\right] = I\left[\tau(P_1)\right] e^{-\tau(P_2)+\tau(P_1)} + \int_{\tau(P_1)}^{\tau(P_2)} dt B(t) e^{\tau-\tau(P_2)}$$

$$= I\left[\tau(P_1)\right] e^{-\tau(P_1,P_2)} + \int_{\tau(P_1)}^{\tau(P_2)} d\tau B(\tau) e^{-\tau(P,P_2)}$$

- But what does this equation tell us about the physics of the problem?

Physical Description



Solution for Zero Scattering

- Break up the path from P_1 to P_2 into small elements δs with optical depths $\delta\tau$
- When σ_n is zero then ε_n is equal to 1
- Hence $\delta\tau B$ is the blackbody emission from the element ds
- The intensity at P_1 is $I[\tau(P_1)]$
- This intensity will be absorbed as it moves from P_1 to P_2 , and the intensity at P_2 will be

$$I[\tau(P_1)]exp[-(\tau(P_2)- \tau(P_1)]$$

Solution for Zero Scattering

- Now consider each small element P with a $\Delta\tau$, with an optical depth τ
- Emission from each element is $B\Delta\tau$
- The amount of this radiation that reaches P_2 is $B\,\Delta\,\tau\,exp[-\tau(P,P_2)]$ where τ is the optical depth between P and P_2
- Hence the total amount of radiation reaching P_2 from all elements is

$$\sum_0^n \Delta\tau B(\tau) e^{-\tau(P,P_2)} \equiv \int_{\tau(P_1)}^{\tau(P_2)} B(\tau) e^{-\tau(P,P_2)} \delta\tau$$

Isothermal Medium – Arbitrary Geometry

Redefine the origin such that $\tau(\mathrm{P}_1) = 0$, then

$$I[\tau(P_2)] = I[\tau(P_1)]e^{-\tau(P_2)} + B\int_0^{\tau(P_2)} dte^{-(\tau(P_2)-\tau)}$$

$$= I[0]e^{-\tau(P_2)} + B\left[1 - e^{-\tau(P_2)}\right]$$

If the medium is optically thin, i.e. $\tau(P_2) << 1$ then the second term becomes B τ(P2).

If there is no absorption or scattering then $\tau=0$ and the intensity in any direction is a constant, i.e. $I[\tau(P_2)]=I[\tau(P_1)]$

Isothermal Medium – Arbitrary Geometry

If we consider the case when τ>>1 then the total intensity is equal to B(T). In this case the medium acts like a blackbody in all frequencies, i.e. is in a state of thermodynamic equilibrium.

If ones look toward the horizon then in a homogeneous atmosphere the atmosphere has a constant temperature. Hence the observed intensity is also blackbody

Zero Scattering in Slab Geometry

- Most common geometry in the theory of radiative transfer is a plane-parallel medium or a slab.
- The vertical optical path (optical depth) is given the symbol τ as distinct from the slant optical path τ_s
- Using z as altitude $\tau(z) = \tau_s\,|\cos\theta| = \tau_s\,\mu$
- The optical depth is measured along the vertical downward direction, i.e. from the 'top' of the medium

Half-range Intensities

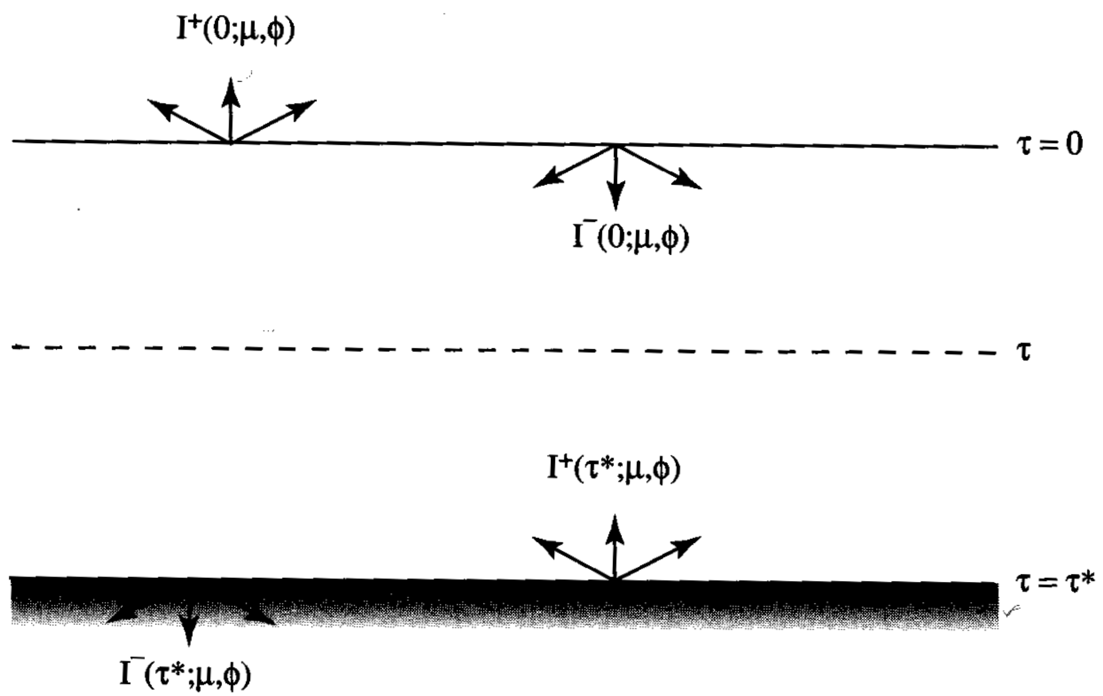


Figure 5.10 Half-range intensities in a slab geometry. The optical depth variable τ is measured downward from the "top" of the medium ($\tau = 0$) to the "bottom" ($\tau = \tau^*$). μ is equal to the absolute value of the cosine of the angle θ, the polar angle of the propagation vector $\hat{\Omega}$.

Half-Range Quantities in Slab Geometry

- The *half-range intensities* are defined by:

$$I_\nu^+(\tau,\theta,\phi) \equiv I_\nu(\tau,\theta \le \pi/2,\phi)$$
$$I_\nu^-(\tau,\theta,\phi) \equiv I_\nu(\tau,\theta > \pi/2,\phi)$$

- Note that the negative direction is for the downward flux,

Half-Range Quantities in Slab Geometry

- The radiative flux is also defined in terms of half-range quantities.

$$F_\nu^+ = \int_+ d\omega \cos\theta I_\nu^+(\hat{\Omega}) = \int_0^{2\pi} d\phi \int_0^{\pi/2} d\theta \sin\theta \cos\theta I_\nu^+(\tau,\theta,\phi)$$

$$= \int_0^{2\pi} d\phi \int_0^1 d\mu \mu I_\nu^+(\tau,\mu,\phi)$$

$$F_\nu^- = -\int_+ d\omega \cos\theta I_\nu^-(\hat{\Omega}) = -\int_0^{2\pi} d\phi \int_{\pi/2}^{\pi} d\theta \sin\theta \cos\theta I_\nu^-(\tau,\theta,\phi)$$

$$= \int_0^{2\pi} d\phi \int_0^1 d\mu \mu I_\nu^-(\tau,\mu,\phi)$$

Half Range Quantities

- In the limit of no scattering the radiative transfer equations for the half-range intensities become

$$\mu \frac{dI_\nu^+(\tau,\mu,\phi)}{d\tau} = I_\nu^+(\tau,\mu,\phi) - B(\tau)$$

$$-\mu \frac{dI_\nu^-(\tau,\mu,\phi)}{d\tau} = I_\nu^-(\tau,\mu,\phi) - B(\tau)$$

Formal Solution in Slab Geometry

- Choose the integrating factor e $^{\tau/\mu}$, for the first equation, then

$$\frac{d}{d\tau}\left(I_\nu^- e^{\tau/\mu}\right)=\left(\frac{dI_\nu^-}{d\tau}+\frac{1}{\mu}I_\nu^-\right)e^{\tau/\mu}=\frac{B_\nu(\tau)}{\mu}e^{\tau/\mu}$$

- This represents a downward beam so we integrate from the "top" of the atmosphere ($\tau=0$) to the bottom ($\tau=\tau^*$).

Slab geometry

$$\int_0^{\tau^*} d\tau' \frac{d}{d\tau'}\left(I_\nu^- e^{\tau'/\mu}\right) = I_\nu^-(\tau^*,\mu,\phi)e^{\tau^*/\mu} - I_\nu^-(0,\mu,\phi)$$

$$= \int_0^{\tau^*} \frac{d\tau'}{\mu} e^{\tau'/\mu} B_\tau(\tau')$$

or

$$I_\nu^-(\tau^*,\mu,\phi) = I_\nu^-(0,\mu,\phi)e^{-\tau^*/\mu} + \int_0^{\tau^*} \frac{d\tau'}{\mu} B_\nu(\tau') e^{-(\tau^*-\tau')/\mu}$$

Slab Geometry

- For an interior point, $\tau < \tau^*$, we integrate from 0 to τ. The solution is easily found by replacing τ^* with τ

$$I_\nu^-(\tau,\mu,\phi) = I_\nu^-(0,\mu,\phi)e^{-\tau/\mu} + \int_0^\tau \frac{d\tau'}{\mu} B_\nu(\tau')e^{-(\tau-\tau')/\mu}$$

Mid-Term Exam
March 15