# AOSC 621

## Phase Function & RTE Solution

# *Solution of the RTE with scattering*

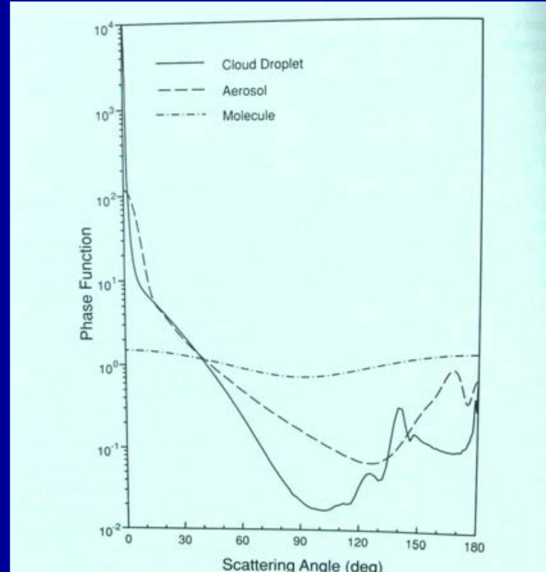



**Figure 3.13** Normalized phase functions for cloud droplets (~10 μm), aerosols (~1 μm), and molecules (~$10^{-4}$ μm) illuminated by a visible wavelength of 0.5 μm, computed from the Lorenz–Mie theory.

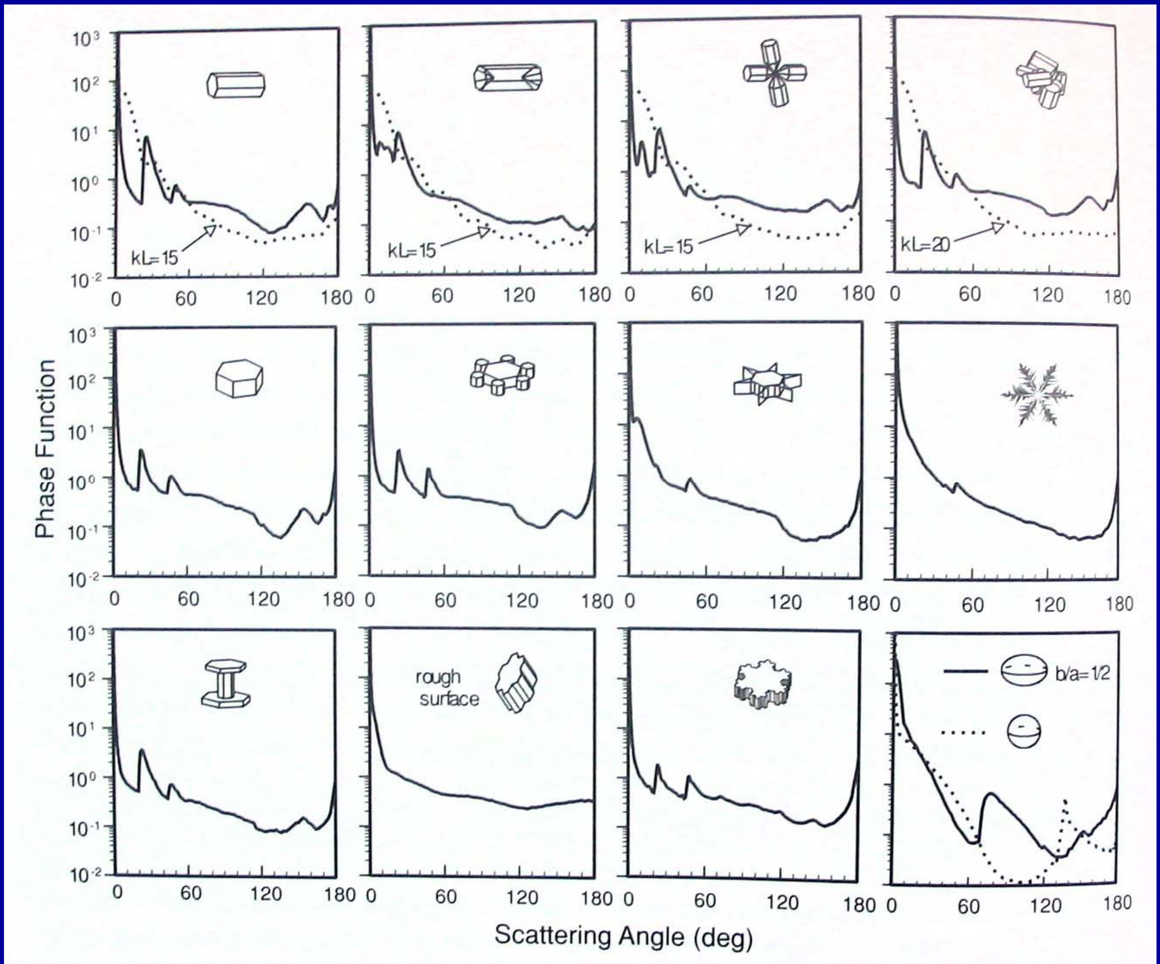
*Solving equation that includes scattering is difficult, because it is difficult to evaluate the integral in the scattering source term* → $S_{scatt}$

*The problem is that*
- *I is peaked (huge peak at sun)*
- *P is peaked (diffraction peak)*

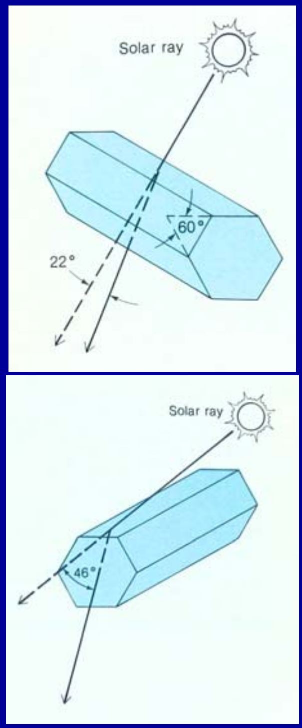
$$\frac{dI}{d\tau_s} = \mu \frac{dI}{d\tau} = -I + (1-a)\cdot B + a \cdot \underbrace{\frac{1}{4\pi}\int_{4\pi} P(\Omega',\Omega)\cdot I(\Omega')\cdot d\Omega'}_{S_{scatt}}$$

*Peakiness is problem because while smooth functions can be integrated using a coarse angular resolution (e.g., using few Legendre polynomial terms), peaky functions need a higher resolution (more terms), and this makes the calculations much more lengthy.*

# *Sample ice crystal phase functions*




*22° and 46° halos*



# *Methods for solving the RTE with scattering*

***Approximate methods:***

*2-stream approximation*
*Eddington approximation*
*Integral form of radiative transfer equation*
*Single scattering approximation*

***Accurate methods:***

*Successive orders of scattering*
*Principle of reciprocity*
*Adding-doubling*
*Discrete ordinates*
*Spherical harmonics*

***Most accurate method:***
*Monte Carlo*

# Azimuthal Dependence

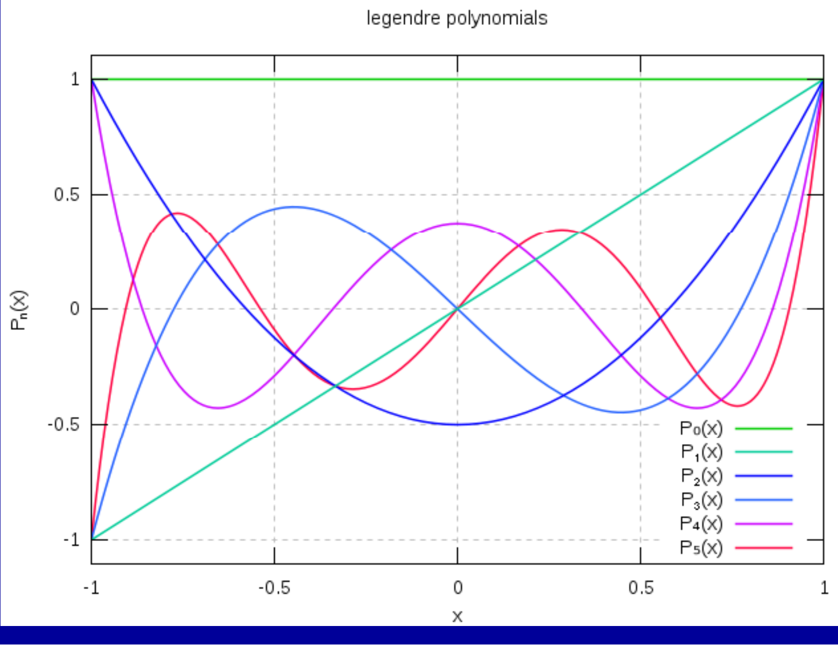
- In slab geometry, the flux and mean intensity depend only on τ and μ. If we need to solve for the intensity or source function, then we need to solve for τ, μ and ϕ. However it is possible to reduce the problem to two variables by introducing a mathematical transformation.

$$p(\tau,\Theta) \approx \sum_{l=0}^{2N-1}(2l+1)\chi(\tau)P_l(\cos\Theta)$$

where $P_l$ is the $l$th *Legendre Polynomial*

# Legendre Polynomial

| n | $P_n(x)$ |
|---|---|
| 0 | $1$ |
| 1 | $x$ |
| 2 | $\frac{1}{2}(3x^2-1)$ |
| 3 | $\frac{1}{2}(5x^3-3x)$ |
| 4 | $\frac{1}{8}(35x^4-30x^2+3)$ |
| 5 | $\frac{1}{8}(63x^5-70x^3+15x)$ |
| 6 | $\frac{1}{16}(231x^6-315x^4+105x^2-5)$ |
| 7 | $\frac{1}{16}(429x^7-693x^5+315x^3-35x)$ |
| 8 | $\frac{1}{128}(6435x^8-12012x^6+6930x^4-1260x^2+35)$ |
| 9 | $\frac{1}{128}(12155x^9-25740x^7+18018x^5-4620x^3+315x)$ |
| 10 | $\frac{1}{256}(46189x^{10}-109395x^8+90090x^6-30030x^4+3465x^2-63)$ |



# Legendre Polynomials

- The Legendre polynomials comprise a natural basis set of orthogonal polynomials over the domain $(0 \le \Theta \le 180)$

- Legendre polynomials are orthogonal

$$\frac{1}{2}\int_{-1}^{+1} du P_l(u) P_k(u) = \frac{1}{2l+1}\delta_{lk}$$

where $\delta_{lk} = 1 \; for \; l = k$, but $= 0$ for $l \neq k$

# *Legendre expansion & Henyey-Greenstein Function*

*Goal: describe phase function (P) using few parameters so that it can be handled easily in equation of radiative transfer*

$$P(\cos\Theta) \approx \sum_{l=0}^{2N-1} (2l+1) \cdot \chi_l \cdot P_l(\cos\Theta)$$

*$P_l$ is $l^{th}$ order Legendre polynomial*
*(function for any x between –1 & 1)*

$$P_l(x) = \frac{1}{2^l \cdot l!} \frac{d^l}{dx^l} (x^2 - 1)^l$$

$$P_0(x) = 1 \quad P_1(x) = x \quad P_2(x) = \frac{1}{2}(3x^2 - 1) \quad P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

*$\chi_l$ is case specific Legendre coefficient, given by*

$$\chi_l \approx \frac{1}{2} \int_{-1}^{1} P(\cos\Theta) \cdot P_l(\cos\Theta) \cdot d(\cos\Theta)$$

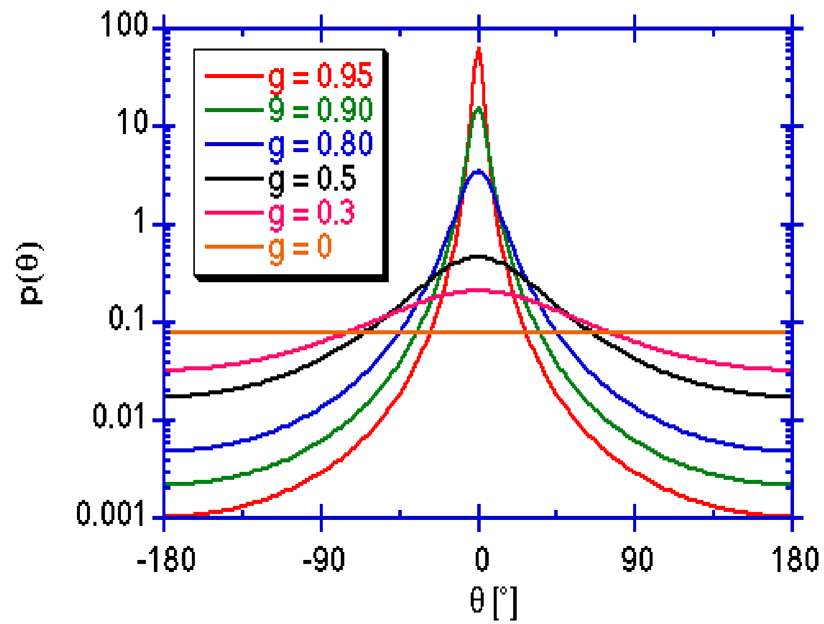
$$\chi_0 = 1$$

$$\chi_1 \approx \frac{1}{2} \int_{-1}^{1} P(\cos\Theta) \cdot \cos\Theta \cdot d(\cos\Theta) = \frac{1}{2} \int_{0}^{180} P(\Theta) \cdot \cos\Theta \cdot d\Theta = g$$

*Simple approximation that uses only three terms to get:*
***Henyey-Greenstein phase function:***

$$P(\Theta) = \frac{1 - g^2}{1 + g^2 - 2g\cos\Theta} \qquad \chi_0 = 1 \qquad \chi_1 = g \qquad \chi_2 = g^2$$

*Henyey and Greenstein (1941) devised an expression which mimics the angular dependence of light scattering by small particles*



# Mie-Debye Phase Function

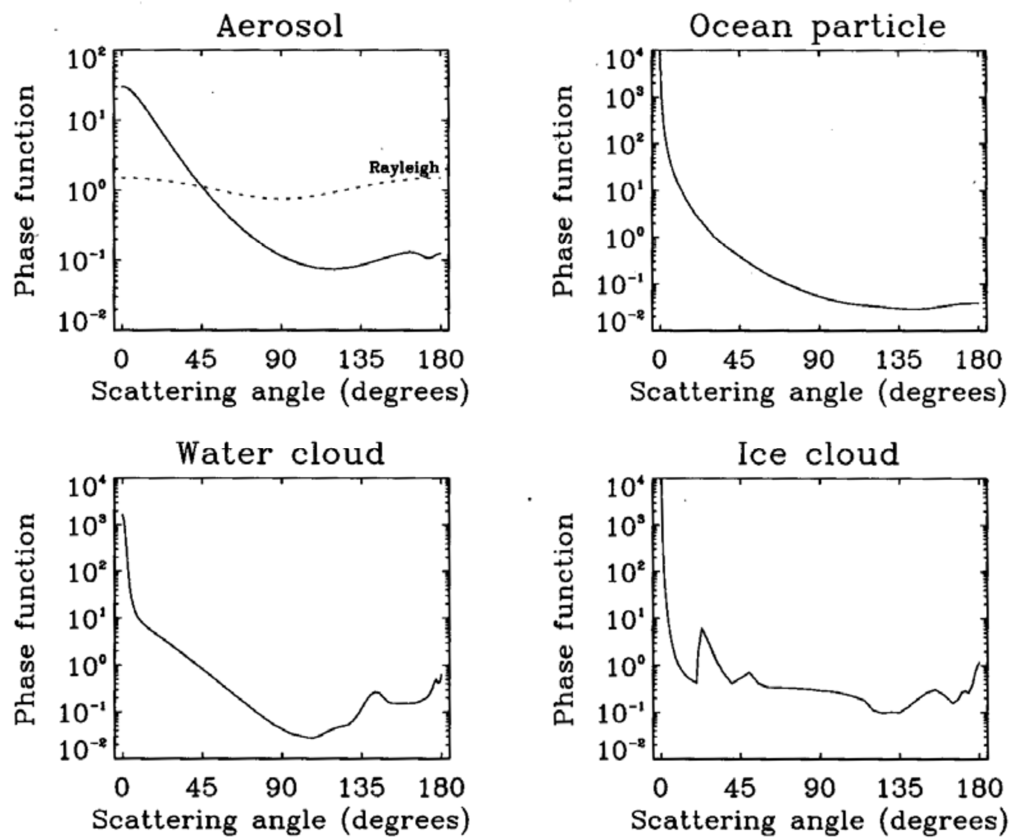



**Figure 6.3** Illustration of phase functions occurring in planetary media. Shown are phase functions for: molecular (Rayleigh) scattering and aerosol particles (upper left); hydrosols (upper right); cloud droplets (lower left); and ice crystals (lower right).

# Azimuthal Dependence

- The first moment of the phase function is commonly denoted by the symbol $g=\chi_1$.
- This represents the degree of assymetry of the angular scattering and is called the *assymetry factor*. Special values of *g* are
- When g=0 - isotropic scattering
- When g=-1 - complete backscattering
- When g=+1- complete forward scattering

# Azimuthal Dependence

- We can now expand the phase function

$$p(\cos\Theta) = p(u',\phi';u,\phi)$$

$$p(u',\phi';u,\phi) = \sum_{m=0}^{2N-1} p^m(u',u)\cos m(\phi'-\phi) \quad \text{where}$$

$$p^m(u',u) = (2-\delta_{0m})\sum_{l=m}^{2N-1}(2l-1)\chi_l \mathrm{A}_l^m(u')\mathrm{A}_l^m(u)$$

# Azimuthal Dependence

- This expansion of the phase function is essentially a Fourier cosine series, and hence we should be able to expand the intensity in a similar fashion.

$$I(\tau,u,\phi)=\sum_{m=0}^{2N-1} I^m(\tau,u)\cos m(\phi_0-\phi)$$

- We can now write a radiative transfer equation for each component

# Azimuthal Dependence

$$u\frac{dI^m(\tau,u,\phi)}{d\tau} = I^m(\tau,u,\phi)$$

$$-\frac{a}{4\pi}\int_0^{2\pi} d\phi' \int_{-1}^{+1} du'\, p^m(u'.\phi';u,\phi) I^m(\tau,u',\phi')$$

$$-X_0^m(\tau,u)e^{-\tau/\upsilon_0}$$

$$-(1-a)B(\tau)\delta_{0m} \quad (\mathrm{m} = 0,1,2,....,2\mathrm{N}\ \text{-}1)$$

$$\text{where } X_0^m(\tau,u) = \frac{a}{4\pi}F^S(2-\delta_{0m})p^m(\tau,-\mu_0,u)$$

# Examples of Phase Functions

- Rayleigh Phase Function. If we assume that the molecule is isotropic, and the incident radiation is unpolarized then the normalised phase function is:

$$p_{RAY}(\cos\Theta) = \frac{3}{4}(1+\cos^2\Theta)$$

$$p_{RAY}(u'',\phi'';u,\phi) =$$

$$\frac{3}{4}\left[1+u''^2 u^2 + (1-u''^2)(1-u^2)\cos^2(\phi''-\phi)\right]$$

$$+2u''u(1-u''^2)^{1/2}(1-u^2)^{1/2}\cos(\phi''-\phi)$$

# Rayleigh Phase Function

- The azimuthally averaged phase function is

$$p_{RAY}(u',u) = \frac{1}{2\pi}\int_0^{2\pi} d\phi' \, p_{RAY}(u',\phi';u,\phi)$$

$$= \frac{3}{4}\left[1 + u'^2 u^2 + \frac{1}{2}(1-u'^2)(1-u^2)\right]$$

- In terms of Legendre polynomials

$$p_{RAY}(u',u) = 1 + \frac{1}{2} P_2(u) P_2(u')$$

# Rayleigh Phase Function

- The assymetry factor is therefore

$$g = \chi_l = \frac{1}{2}\int_{-1}^{+1} du' \, p_{RAY}(u',u) P_1(u')$$

$$= \frac{1}{2}\int_{-1}^{+1} du' u' \, p_{RAY}(u',u) = 0$$

# Mie-Debye Phase Function

- Scattering of solar radiation by large particles is characterized by forward scattering with a diffraction peak in the forward direction
- Mie-Debye theory - mathematical formulation is complete. Numerical implementation is challenging
- Scaling transformations

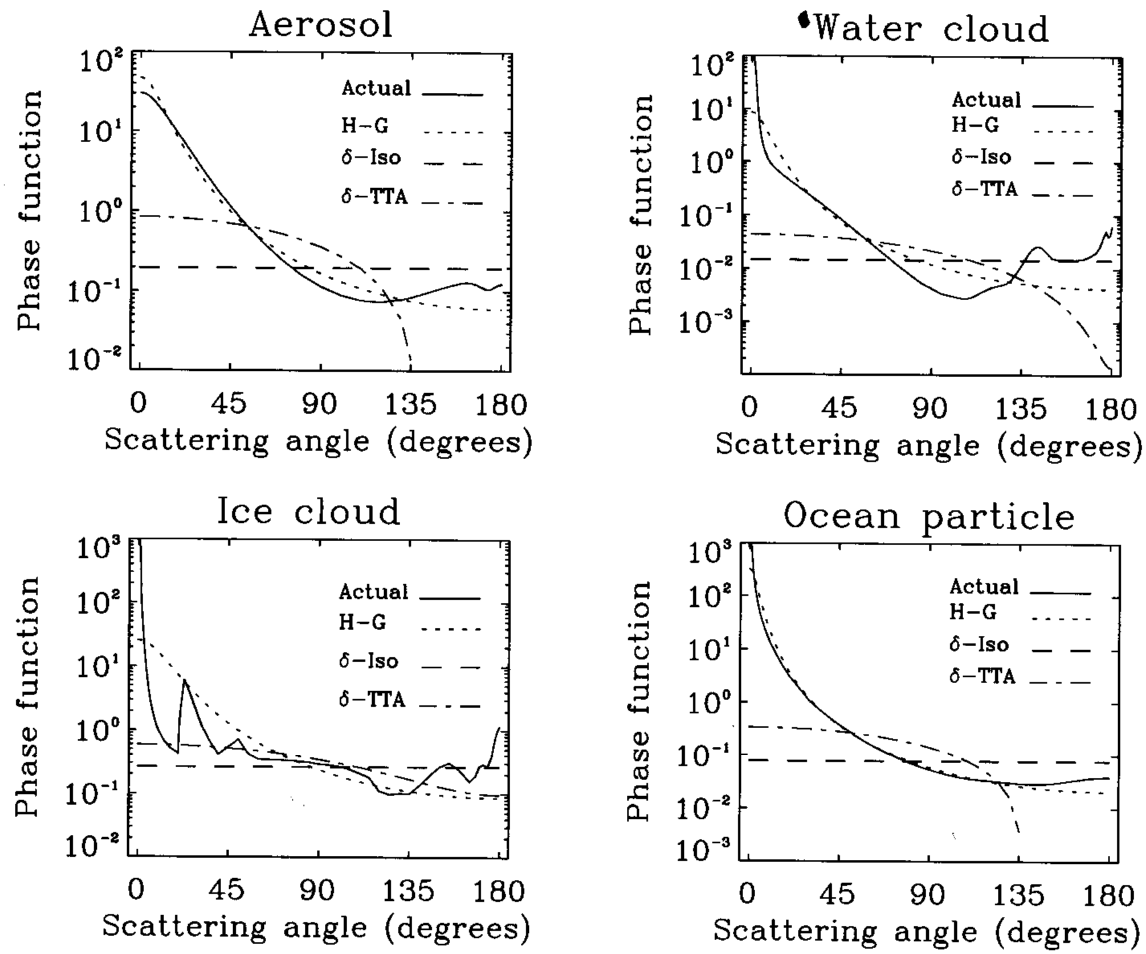



**Figure 6.4** Illustration of actual and $\delta$-$N$ scaled phase functions for aerosol particles, water cloud droplets, ice particles, and hydrosols.

# Scaling Transformations

- The examples shown of the phase function versus the scattering angle all show a strong forward peak. If we were to plot the phase function versus the cosine of the scattering angle - the unit actually used in radiative transfer- then the forward peak becomes more pronounced.
- Approaches a delta function
- Can treat the forward peak as an unscattered beam, and add it to the solar flux term.

- Then the remainder of the phase function is expanded in Legendre Polynomials.

$$\hat{p}_{\delta-N}(\cos\Theta) = \hat{p}_{\delta-N}(u',\phi':u,\phi)$$

$$= 2f\delta(1-\cos\Theta) + (1-f)\sum_{l=0}^{2N-1}(2l+1)\hat{\chi}_l P_l(\cos\Theta)$$

- This is known as the δ–N approximation
- There are simpler approximations

# The δ–Isotropic Approximation

- The crudest form is to assume that, outside of the forward peak, the remainder of the phase function is a constant, Basically this assumes isotropic scattering outside of the peak. The azimuthally averaged phase function becomes

$$\hat{p}_{\delta-ISO}(u',u) = 2f\delta(u'-u) + (1-f)$$

- When this phase function is substituted into the azimuthally averaged radiative transfer equation we get:

# The δ–Isotropic Approximation

$$u\frac{dI(\tau,u)}{d\tau}=I(\tau,u)-\frac{a}{2}\int_{-1}^{1}du'\,p(u',u)I(\tau,u')$$

$$=I(\tau,u)-afI(\tau,u)-\frac{a(1-f)}{2}\int_{-1}^{1}du'\,I(\tau,u')$$

or

$$u\frac{dI(\hat{\tau},u)}{d\hat{\tau}}=I(\hat{\tau},u)-\frac{\hat{a}}{2}\int_{-1}^{1}du'\,I(\hat{\tau},u')$$

where

$$d\hat{\tau}=(1-af)d\tau \text{ and } \hat{a}=\frac{(1-f)a}{1-af}$$

# The δ–Isotropic Approximation

$f$ the strength of the forward scattering peak is given by the relationship

$$\chi_1 = \frac{1}{2}\int_{-1}^{1} du' u' p_{\delta-ISO}(u', u) = f$$

# The δ-Two-Term Approximation

- A better approximation results by representing the remainder of the phase function by two terms (setting N=1 in the full expansion). We now get:

$$\hat{p}_{\delta-TTA}(u',u) =$$

$$2f\delta(u'-u) + (1-f)\sum_{l=0}^{l=1}(2l+1)\chi_l P_l(u')P_l(u)$$

# The δ-Two-Term Approximation

- Substituting into the azimuthally averaged radiative transfer equation:

$$u \frac{dI(\hat{\tau}, u)}{d\hat{\tau}} = I(\hat{\tau}, u)$$

$$-\frac{a}{2} \sum_{l=0}^{l=1} (2l+1)\hat{\chi}_l P_l(u) \int_{-1}^{1} du' P_l(u) I(\hat{\tau}, u')$$

where

$$\hat{\chi}_l = \hat{g} = \frac{\chi_l - f}{1-f} = \frac{g-f}{1-f} \text{ and } f = \chi_2$$