AOSC 621

Lesson 12: Prototype of Radiative Transfer Solution

Prototype problems in Radiative Transfer Theory

We will now study a number of standard radiative transfer problems. Each problem assumes a slab geometry and an optically uniform (homogeneous) medium. The radiation is monochromatic and unpolarized. Complete specification of each problem requires five input variables

Input Variables to a RTE

(1) τ: the vertical optical depth

(2) $S(\tau,\Omega)$: internal or external source function

(3) $p(\Omega',\Omega)$: the phase function

(4) *a* : the single scattering albedo

(5) $\rho(-\Omega',\Omega)$ – the bidirectional reflectance function of the surface

- For a Lambert surface ρ is a constant
- For $\tau^* \rightarrow \infty$ then $S(\tau)e^{t} \rightarrow 0$

Output Variables of a RTE

- The analytic or numerical solutions provide the following output variables among others

  (1) The reflectance

  (2) The transmittance

  (3) The absorptance

  (4) The emittance

  (5) The internal intensity field

  (6) The heating rate and net flux

Prototype problems

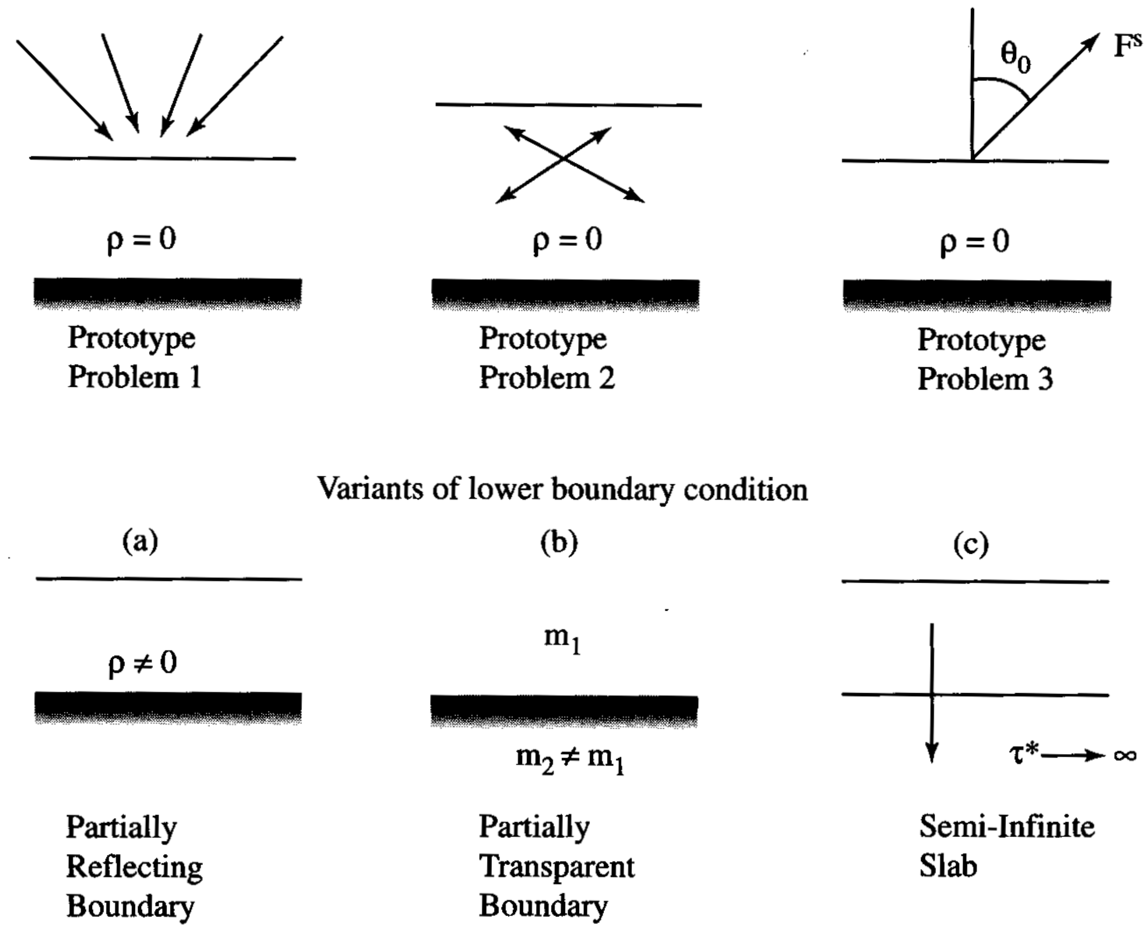


Figure 6.5 Illustration of Prototype Problems in radiative transfer.

- If the source function is known then we may integrate the radiative transfer equation directly, e.g. if scattering is ignored and we are only dealing with thermal radiation.

- This is also true if we can ignore multiple scattering , and consider single scattering approximation

Problem #1 – Uniform Illumination

- The incident field is taken to be constant in the downward direction
- The radiation field depends only on τ and μ
- The source function depends only on τ
- It approximately reproduces the effect of an optically thick cloud overlying an atmosphere
- The source for the diffuse radiation is

$$S^*(\tau) = \frac{a}{4\pi} \int_0^{2\pi} d\phi \int_0^1 d\mu \, I \, e^{-\tau/\mu}$$

Prototype problems

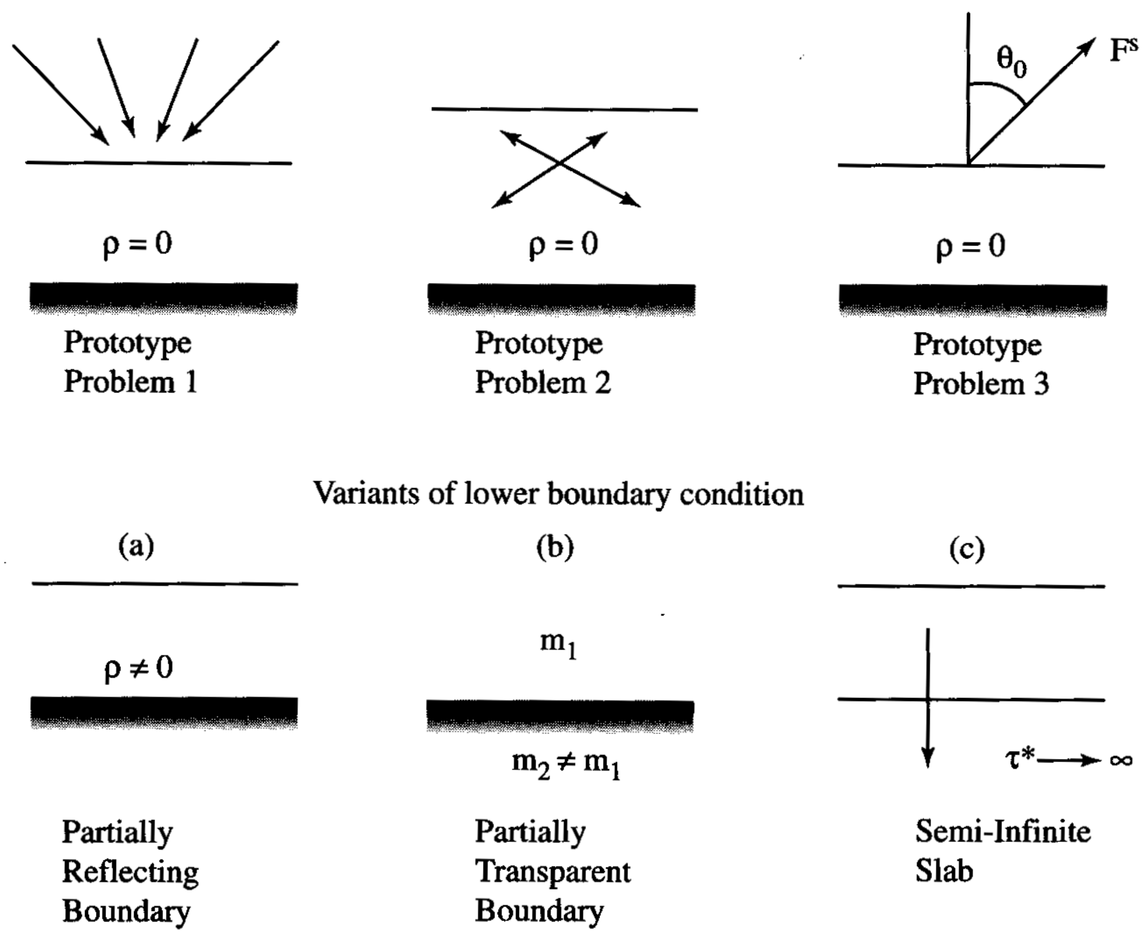


Figure 6.5 Illustration of Prototype Problems in radiative transfer.

Problem #2 Constant Imbedded Source

- For thermal radiation problems the term

$$(1-a)B$$

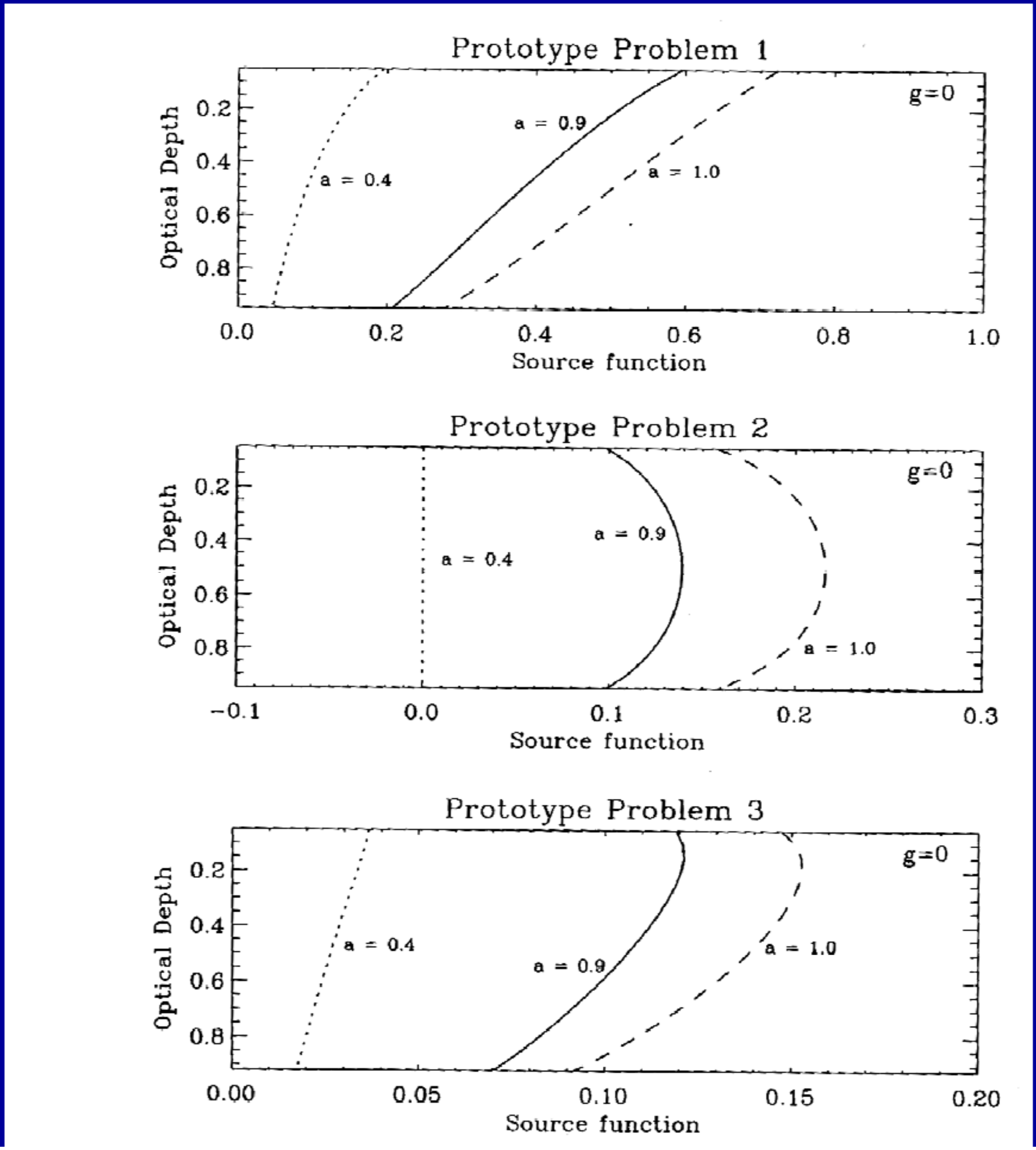
is the driver of the radiation. This is an imbedded source. In general this term is a strong function of frequency and temperature. We will assume that the term is constant with depth.

Problem #3 – Diffuse Reflection Problem

- In this problem we consider collimated incidence and a lower boundary that may be partly reflecting. For shortwave applications the term *(1-a)B* can be ignored. The only term is:

$$S^*(\tau, \pm\mu, \phi) = \frac{aF^S}{4\pi} p(-\mu_0, \phi_0; \pm\mu, \phi) e^{-\tau/\mu_0}$$

Prototype problems



Prototype problems

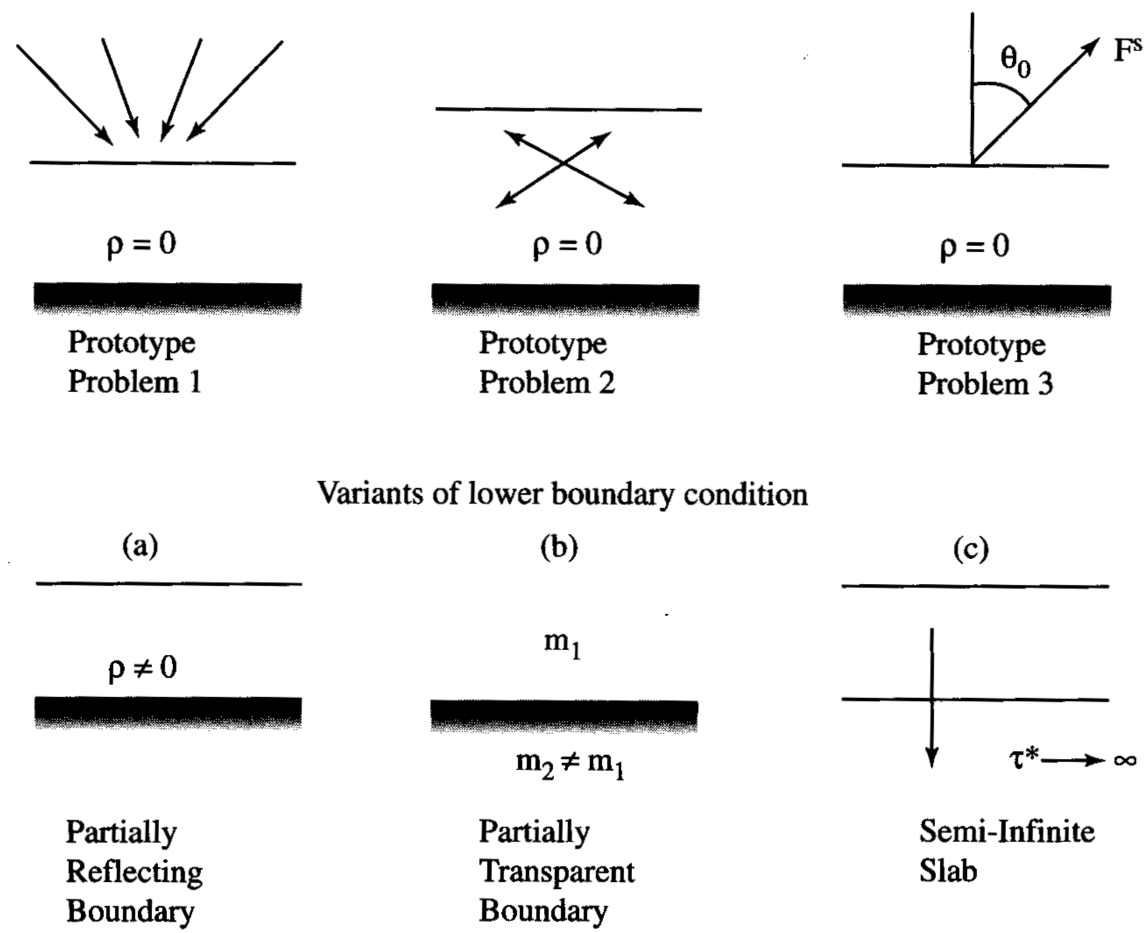


Figure 6.5 Illustration of Prototype Problems in radiative transfer.

Boundary Conditions: Reflecting Surface

- The radiation reflected back from the ground is often comparable to the direct solar radiation.
- First order scattering from this source can be important
- Effects of ground reflection should always be taken into account in any first order scattering calculation.
- For small optical depths the ratio of the reflected component to the direct component can exceed 1.0, even for a surface with a reflectivity of 10%

Boundary Conditions: Reflecting and Emitting surface

- First consider a Lambertian surface (BDRF = μ_L) which also emits thermal radiation with an emittance ε and temperature T_s.
- The upward intensity at the surface is given by :

$$I^+(\tau^*,\mu,\varphi) = \rho_L F_d^-(\tau^*)$$
$$+\rho_L \int_0^{2\pi} d\varphi' \int_0^1 d\mu \mu I^-(0,\mu,\varphi) e^{-\tau^*/\mu} + \varepsilon_s B(T_s)$$

Boundary Conditions: Reflecting and Emitting surface

- The upper and lower boundary conditions for the three prototype problems are:
- Prototype problem 1

$$I^{-}(0, \mu) = \text{a constant} = \text{I}$$

$$I^{+}(\tau^{*}, \mu) = \rho_L \left[F_d^{-}(\tau^{*}) + 2\pi \text{I} E_3(\tau^{*}) \right] + \varepsilon_s B(T_s)$$

where

$$E_n(x) = \int_0^1 d\mu \mu^{n-2} e^{-x/\mu}$$

Prototype problems

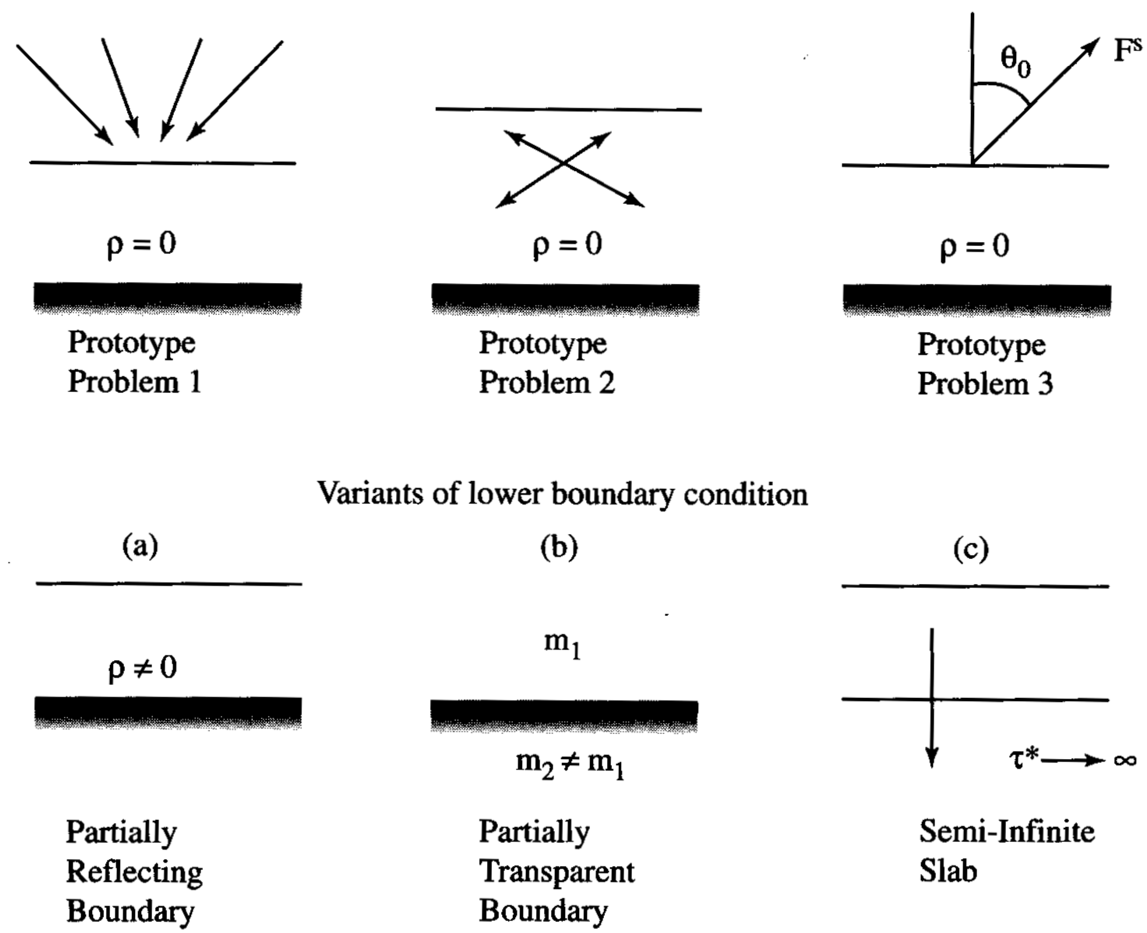


Figure 6.5 Illustration of Prototype Problems in radiative transfer.

Boundary Conditions: Reflecting and Emitting surface

Prototype problem 2

$$I^-(0,\mu) = 0 \qquad I^+(\tau^*,\mu) = \rho_L F_d^-(\tau^*)$$

Prototype problem 3

$$I^-(0,\mu,\varphi) = F^S \delta(\mu - \mu_0)\delta(\varphi - \varphi_0)$$

$$I^+(\tau^*,\mu,\varphi) = \rho_L \left[F_d^-(\tau^*) + \mu_0 F^S e^{-\tau^*/\mu_0} \right]$$

Reciprocity, Duality and Inhomogeneous Media

- The *Reciprocity Principle* states that, in any linear system, the pathways leading from a cause at one point to an effect at another point can equally be traversed in the opposite direction. Hence for the BRDF and flux reflectance:

$$\rho(-\hat{\Omega}',\hat{\Omega}) = \rho(-\hat{\Omega},\hat{\Omega}')$$

$$\rho(-\hat{\Omega}',-2\pi) = \rho(-2\pi,-\hat{\Omega}')$$

Transmittance

- Reciprocity relations also exist for the transmittance.

$$\mathscr{T}(-\hat{\Omega}', \ \hat{\Omega}) = \overline{\mathscr{T}}(+\hat{\Omega}, +\hat{\Omega}')$$
$$\mathscr{T}(-2\pi, \ \hat{\Omega}) = \overline{\mathscr{T}}(+\hat{\Omega}, +2\pi)$$

Previous discussions have been limited to homogeneous atmospheres. However, in general, the reflectance and transmission illuminated from above, are different from those illuminated from below.

Surface Reflection

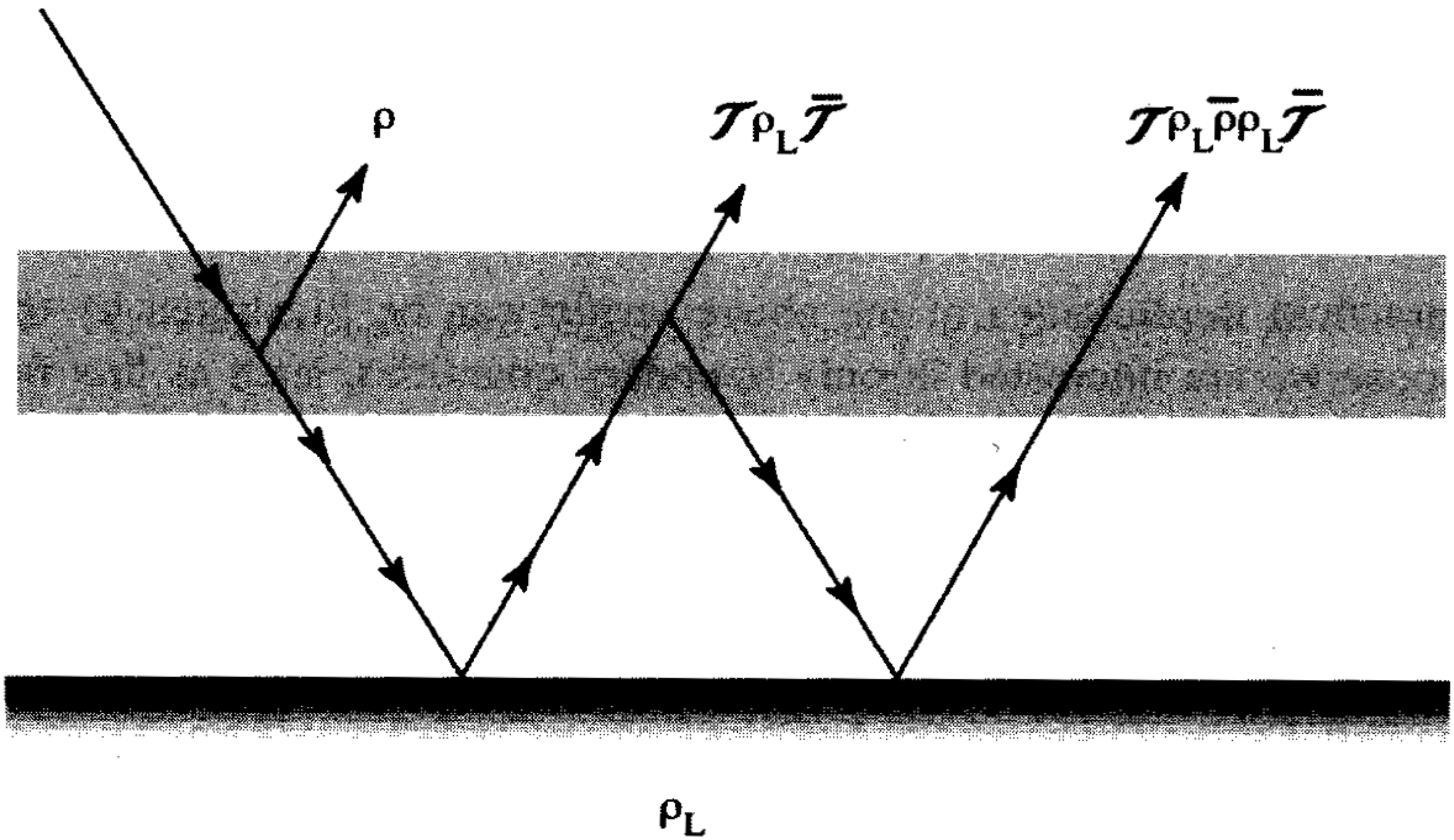


Figure 6.7 Addition of a reflecting surface leads to a geometric (binomial) series.

Surface Reflection

- Consider the effect of a reflecting lower boundary.
- The light is continuously being 'reflected' by the surface and the slab itself. Hence we end up with an infinite series of beams that add up to get

$$\rho(-\hat{\Omega},+2\pi)+\mathscr{T}(-\hat{\Omega},-2\pi)\rho_L\overline{\mathscr{T}}\left\{1+\bar{\rho}\rho_L+(\bar{\rho}\rho_L)^2.....\right\}$$

$$=\rho(-\hat{\Omega},+2\pi)+\frac{\mathscr{T}(-\hat{\Omega},-2\pi)\rho_L\overline{\mathscr{T}}}{1-\bar{\rho}\rho_L}$$

thus

$$\rho_{tot}(-\hat{\Omega},+2\pi,\rho)=\rho(-\hat{\Omega},+2\pi)+\frac{\mathscr{T}(-\hat{\Omega},-2\pi)\rho_L\overline{\mathscr{T}}}{1-\bar{\rho}\rho_L}$$

Surface Reflection

- A similar equation can be derived for the transmittance

$$\mathcal{T}_{tot}(-\hat{\Omega},-2\pi,\rho)=\mathcal{T}(-\hat{\Omega},+2\pi)+\frac{\mathcal{T}(-\hat{\Omega},-2\pi)\rho_L\bar{\rho}}{1-\bar{\rho}\rho_L}$$

$$=\frac{\mathcal{T}(-\hat{\Omega},-2\pi)}{1-\bar{\rho}\rho_L}$$

Separation of the radiation field into orders of scattering

- It is useful when an approximate solution is available for the multiple scattering, for example from the two-stream approximation. In this case the diffuse intensity is given by the sum of the single-scattering and the approximate multiple-scattering contributions
- It serves as a starting point for expanding the radiation field in a sum of contributions from first-order, second-order scattering etc.

Separation of the radiation field into orders of scattering

In slab geometry, the half range intensities are given by

$$\mu \frac{dI^{+}(\tau,\mu,\phi)}{d\tau} = I^{+}(\tau,\mu,\phi) - S^{+}(\tau,\mu,\phi)$$

$$-\mu \frac{dI^{-}(\tau,\mu,\phi)}{d\tau} = I^{-}(\tau,\mu,\phi) - S^{-}(\tau,\mu,\phi)$$

Formal solutions to these equations are a sum of direct (I_S) and diffuse (I_d)

Separation of the radiation field into orders of scattering

$$I^-(\tau,\mu,\phi) = I^-(0,\mu,\phi)e^{-\tau/\mu} \quad (I_S^-)$$

$$+\int_0^\tau \frac{d\tau'}{\mu} e^{-(\tau-\tau')/\mu} S^-(\tau',\mu,\phi) \quad (I_d^-)$$

$$I^+(\tau,\mu,\phi) = I^+(\tau^*,\mu,\phi)e^{-(\tau^*-\tau)/\mu} \quad (I_S^+)$$

$$+\int_\tau^{\tau^*} \frac{d\tau'}{\mu} e^{-(\tau'-\tau)/\mu} S^+(\tau',\mu,\phi) \quad (I_d^+)$$

$$I(\tau,\mu=0,\phi) = S(\tau)$$

Separation of the radiation field into orders of scattering

In the single scattering case the source simplifies to

$$S^{\pm}(\tau',\mu,\phi) = (1-a)B + \frac{aF^S}{4\pi}\, p(-\mu_0,\phi_0;\pm\mu,\phi)e^{-\tau'/\mu_0}$$

substituting these source functions we get

$$I_d^-(\tau,\mu,\phi) = (1-a)B[1-e^{-\tau/\mu}] + \frac{a\mu_0 F^S p(-\mu_0,\phi_0;-\mu,\phi)}{4\pi(\mu_0-\mu)}[e^{-\tau/\mu_0} - e^{-\tau/\mu}]$$

$$I_d^+(\tau,\mu,\phi) = (1-a)B[1-e^{-(\tau^*-\tau)/\mu}] + \frac{a\mu_0 F^S p(-\mu_0,\phi_0;+\mu,\phi)}{4\pi(\mu_0+\mu)}[e^{-\tau/\mu} - e^{-[(\tau^*-\tau)/\mu+\tau^*/\mu_0]}]$$

Separation of the radiation field into orders of scattering

To obtain the total intensity we must introduce two boundary terms, $I^-(0,\mu,\phi) = F^S\delta(\mu-\mu_0)\delta(\phi-\phi_0)$ and $I^+(\tau^*,\mu,\phi) = 0$ (if we assume that the lower boundary is a perfect absorber). Then we get an analytic solution

$$I^-(\tau,\mu,\phi) = I^-(0,\mu,\phi)e^{-\tau/\mu} + I_d^-(\tau,\mu,\phi)$$
$$= F^S e^{-\tau/\mu}\delta(\mu-\mu_0)\delta(\phi-\phi_0) + I_d^-(\tau,\mu,\phi)$$

$$I^+(\tau,\mu,\phi) = I^+(\tau^*,\mu,\phi)e^{-(\tau^*-\tau)/\mu} + I_d^+(\tau,\mu,\phi)$$
$$= I_d^+(\tau,\mu,\phi)$$

Separation of the radiation field into orders of scattering

- Favorable aspects of the single scattering approximation are
- The solution is valid for any phase function
- It is easily generalized to include polarization
- It applies to any geometry as long as τ/μ is replaced with an appropriate expression. For example, in spherical geometry, with $\tau Ch(\mu_0)$ where *Ch* is the Chapman function

Lambda Iteration

- Assume isotropic scattering in a homogeneous atmosphere. Then we can write

$$S(\tau) = (1-a)B(\tau) + S^*(\tau) + \frac{a}{2}\int_0^{\tau^*} d\tau' E_1(|\tau'-\tau|)S\ (\tau')$$

where $$E_n(x) = \int_0^1 d\mu \mu^{n-2} e^{-x/\mu} = \int_1^\infty \frac{dt}{t^n} e^{-t/x}$$

This integral forms the basis for an iterative solution, in which the first order scattering function is used first for *S*.