METO 621

Lesson 12: Two-stream Solution

Two-stream Approximation- Isotropic Scattering

- Although anisotropic scattering is more realistic, first let's look at isotropic scattering i.e. p=1
- The radiative transfer equations are

$$\mu \frac{dI^+(\tau,\mu)}{d\tau} = I^+(\tau,\mu) - \frac{a}{2}\int_0^1 d\mu' I^+(\tau,\mu') - \frac{a}{2}\int_0^1 d\mu' I^-(\tau,\mu') - (1-a)B$$

Two-stream Approximation- Isotropic Scattering

$$-\mu\frac{dI^{-}(\tau,\mu)}{d\tau}=I^{-}(\tau,\mu)-\frac{a}{2}\int_{0}^{1}d\mu' I^{+}(\tau,\mu')$$

$$-\frac{a}{2}\int_{0}^{1}d\mu' I^{-}(\tau,\mu')-(1-a)B$$

- In the two-stream approximation we replace the angular dependent quantities I by their averages over each hemisphere. This leads to the following pair of coupled differential equations

Two-stream Approximation- Isotropic Scattering

$$\overline{\mu}^{+}\frac{dI^{+}(\tau)}{d\tau}=I^{+}(\tau)-\frac{a}{2}I^{+}(\tau)-\frac{a}{2}I^{-}(\tau)-(1-a)B$$

$$-\overline{\mu}^{-}\frac{dI^{+}(\tau)}{d\tau}=I^{-}(\tau)-\frac{a}{2}I^{+}(\tau)-\frac{a}{2}I^{-}(\tau)-(1-a)B$$

If the medium is homogeneous then *a* is constant. One can now obtain analytic solutions to these equations. $\overline{\mu}$ in the above equations is the cosine of the average polar angle. It generally differs in the two hemispheres

Two-stream Approximation

- The expressions for the source function, flux and heating rate are

$$S(\tau) = \frac{a}{2}\int_0^1 d\mu[I^+(\tau,\mu) + I^-(\tau,\mu)] + (1-a)B$$

$$\approx \frac{a}{2}[I^+(\tau,\mu) + I^-(\tau,\mu)] + (1-a)B$$

$$F(\tau) = 2\pi\int_0^1 \mu d\mu[I^+(\tau,\mu) + I^-(\tau,\mu)]$$

$$\approx 2\pi[\overline{\mu}^+ I^+(\tau,\mu) + \overline{\mu}^- I^-(\tau,\mu)] \text{ and}$$

$$\mathcal{H} = \frac{\delta F}{\delta z} \approx 2\pi\alpha[I^+(\tau,\mu) + I^-(\tau,\mu)] - 4\pi\alpha B$$

The Mean Inclination

$\bar{\mu}$, the mean inclination, could be defined as the intensity weighted mean

$$\bar{\mu}^{\pm} = \frac{2\pi \int_0^1 d\mu \mu I^{\pm}(\tau, \mu)}{2\pi \int_0^1 d\mu I^{\pm}(\tau, \mu)} = \frac{F^{\pm}}{2\pi I^{\pm}}$$

But, of course, if we knew how the intensity varied with τ, μ

we have already solved the problem. Unfortunately there is no magic prescription. In general, the value of the average $\bar{\mu}$ will vary with the optical depth and have a different value in each hemisphere.

The Mean Inclination

- If the radiation is isotropic then the average μ is equal to 0.5 in both hemispheres. If the intensity distribution is approximately linear in μ then the average is 0.666.
- We could also use the root-mean-square value

$$\bar{\mu} \equiv \mu_{rms} = \sqrt{(\mu^2)} = \sqrt{\frac{\int_0^1 d\mu \mu^2 I(\tau,\mu)}{\int_0^1 d\mu I(\tau,\mu)}}$$

Solving the Two-Stream Equations

- Let’s first ignore the thermal emission term (First prototype problem)
- Add and subtract the two equations

$$\bar{\mu}^{+}\frac{dI^{+}(\tau)}{d\tau}=I^{+}(\tau)-\frac{a}{2}I^{+}(\tau)-\frac{a}{2}I^{-}(\tau)-(1-a)B$$

$$-\bar{\mu}^{-}\frac{dI^{+}(\tau)}{d\tau}=I^{-}(\tau)-\frac{a}{2}I^{+}(\tau)-\frac{a}{2}I^{-}(\tau)-(1-a)B$$

We get

$$\bar{\mu}\;\frac{d(I^{+}-I^{-})}{d\tau}=(1-a)(I^{+}+I^{-})$$

$$\bar{\mu}\;\frac{d(I^{+}+I^{-})}{d\tau}=(I^{+}-I^{-})$$

Solving the Two-Stream Equations

- Differentiating the second equation with respect to τ and substituting for $d(I^+-I^-)/d\tau$ from the first equation we get

$$\frac{d^2(I^+ + I^-)}{d\tau^2} = \frac{(1-a)}{\overline{\mu}^2}(I^+ + I^-)$$

- Similarly, differentiate the first equation, and substitute for $d(I^+-I^-)/d\tau$

$$\frac{d^2(I^+ - I^-)}{d\tau^2} = \frac{(1-a)}{\overline{\mu}^2}(I^+ - I^-)$$

Solving the Two-Stream Equations

- We have the same differential equation to solve for both quantities. Calling the unknown *Y*, we obtain a simple second order *diffusion equation.*

$$\frac{d^2Y}{d\tau^2} = \Gamma^2 Y \quad \text{where} \quad \Gamma \equiv (\sqrt{1-\mathrm{a}})/\bar{\mu}$$

- for which the general solution is a sum of positive and negative exponentials

$$Y = A' e^{\Gamma'\tau} + B' e^{-\Gamma'\tau}$$

- *A'* and *B'* are arbitrary constants to be determined

Prototype Problem I

- Since the sum and differences of the two intensities are both expressed as sums of exponentials, each intensity component must also be expressed in the same way.

$$I^+(\tau) = Ae^{\Gamma\tau} + Be^{-\Gamma\tau}, I^-(\tau) = Ce^{\Gamma\tau} + De^{-\Gamma\tau}$$

- where *A*, *B*, *C*.and *D* are additional arbitrary constants.

- We now introduce boundary conditions at the top and bottom of the medium. For prototype problem 1 these are

$$I^-(\tau = 0) = \text{constant} \qquad I^+(\tau^*) = 0$$

Prototype problems

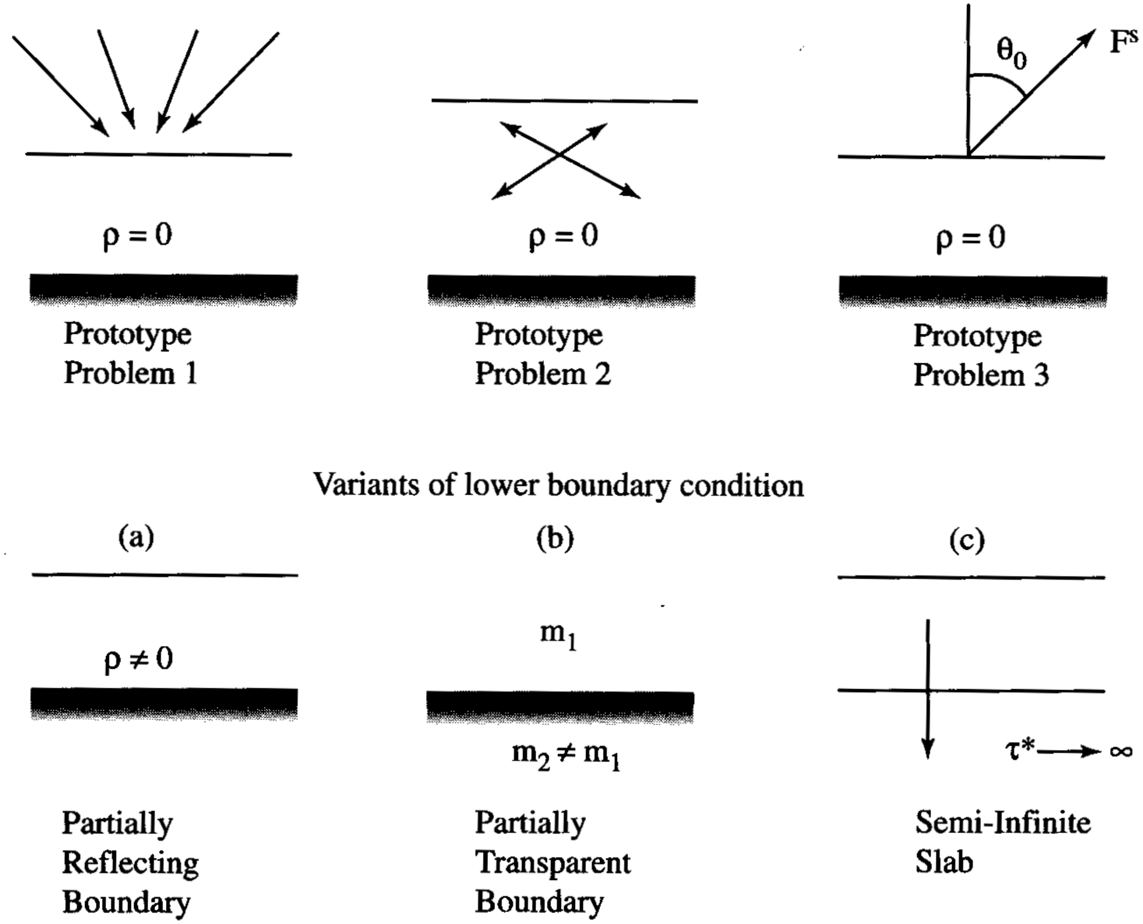


Figure 6.5 Illustration of Prototype Problems in radiative transfer.

Prototype Problem I

- The equation shows four constants of integration, but in fact only two are independent.

$$\frac{C}{A} = \frac{B}{D} = \frac{a}{2 - a + 2\overline{\mu}\Gamma} = \frac{1 - \overline{\mu}\Gamma}{1 + \overline{\mu}\Gamma} = \rho_\infty$$

the solutions are

$$I^+(\tau) = \frac{\bar{I}\rho_\infty}{\mathcal{D}}[e^{\Gamma(\tau^* - \tau)} - e^{-\Gamma(\tau^* - \tau)}]$$

$$I^-(\tau) = \frac{\bar{I}}{\mathcal{D}}[e^{\Gamma(\tau^* - \tau)} - \rho_\infty^2 e^{-\Gamma(\tau^* - \tau)}]$$

where $\mathcal{D} \equiv e^{\Gamma\tau^*} - \rho_\infty^2 e^{-\Gamma\tau^*}$

Two stream solution for uniform illumination (Problem 1)

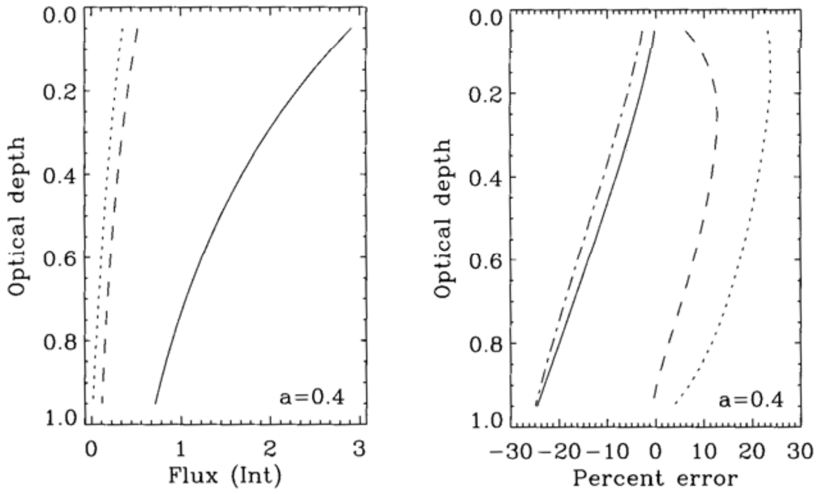
Parameters *I*=1.0, τ*=1.0, a=0.4

μ=0.5 , p=1

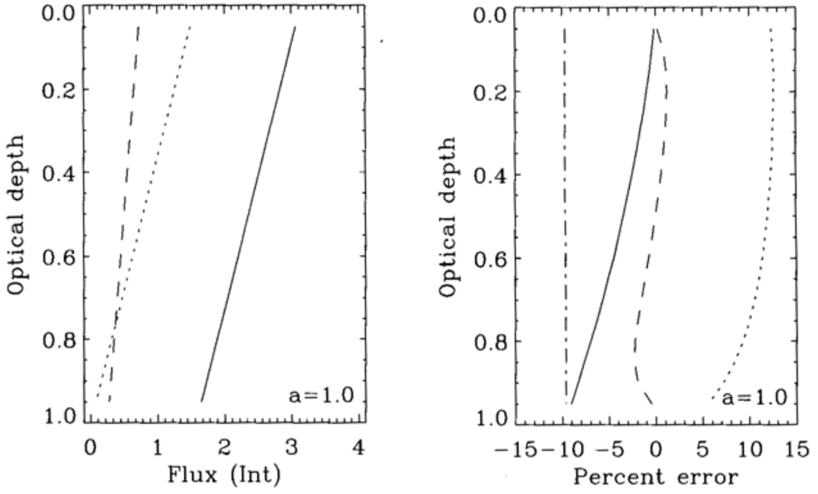
Downward flux = solid line

Upward flux = dotted line

Mean intensity = dashed line



Parameters as above except a=1.0



Prototype problem 2

- Consider the only source of radiation is thermal emission within the slab. The two stream equations are

$$\overline{\mu}^{+}\frac{dI^{+}(\tau)}{d\tau} = I^{+}(\tau) - \frac{a}{2}I^{+}(\tau) - \frac{a}{2}I^{-}(\tau) - (1-a)B$$

$$-\overline{\mu}^{-}\frac{dI^{-}(\tau)}{d\tau} = I^{-}(\tau) - \frac{a}{2}I^{+}(\tau) - \frac{a}{2}I^{-}(\tau) - (1-a)B$$

$$\text{with the boundary conditions } I^{-}(0) = I^{+}(\tau^{*}) = 0$$

Prototype problems

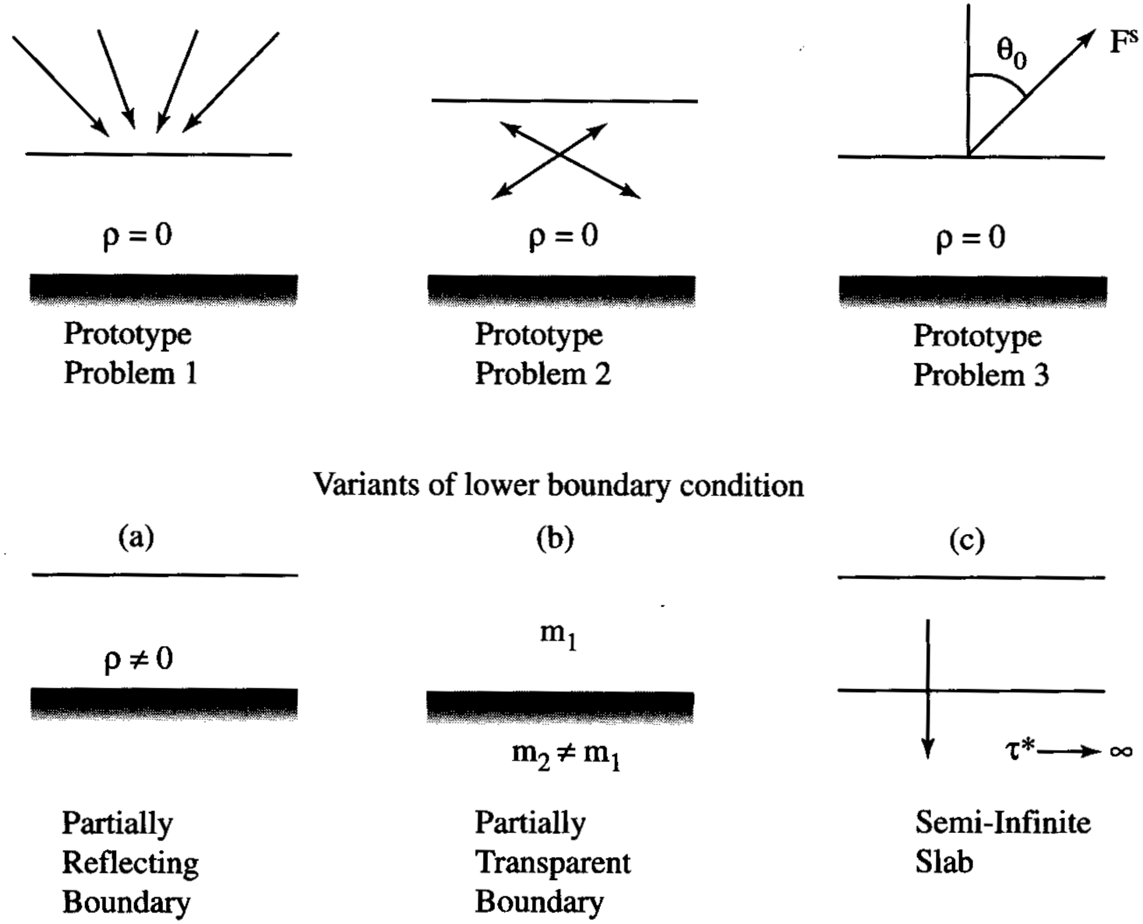


Figure 6.5 Illustration of Prototype Problems in radiative transfer.

Prototype problem 2

- Solving these simultaneous equation starts by seeking the homogeneous solution, and then a particular equation that satisfies the whole equation – using the boundary conditions. We get

$$I^+(\tau) = \frac{B}{\mathcal{D}}\left\{\rho_\infty^2 e^{-\Gamma\tau} - e^{\Gamma\tau} + \rho_\infty[e^{-\Gamma(\tau^*-\tau)} - e^{\Gamma(\tau^*-\tau)}]\right\} + B$$

$$I^-(\tau) = \frac{B}{\mathcal{D}}\left\{\rho_\infty^2 e^{-\Gamma(\tau^*-\tau)} - e^{\Gamma(\tau^*-\tau)} + \rho_\infty[e^{-\Gamma\tau} - e^{\Gamma\tau}]\right\} + B$$

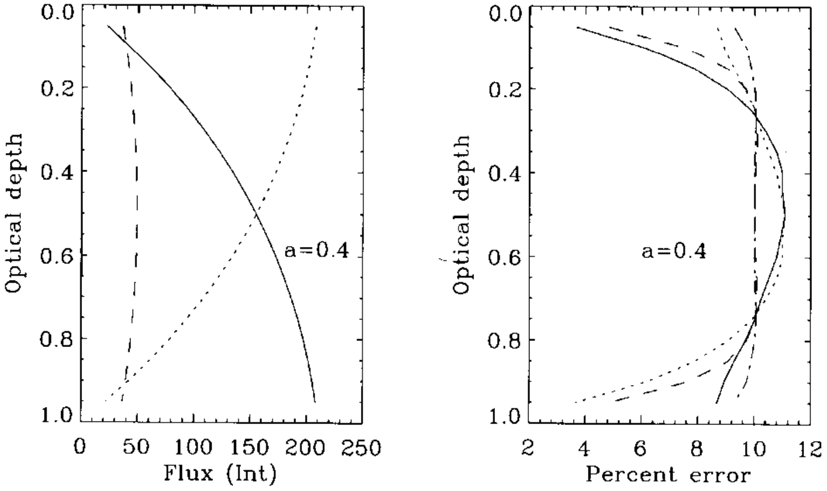
Two-stream solution for an imbedded source

Parameters *B=100*, τ*=1.0, a=0.4, μ=0.5 , p=1

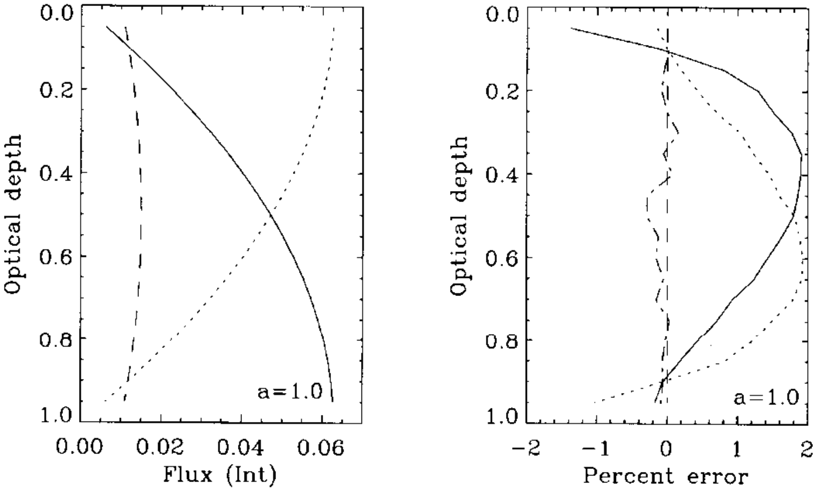
Downward flux = solid line

Upward flux = dotted line

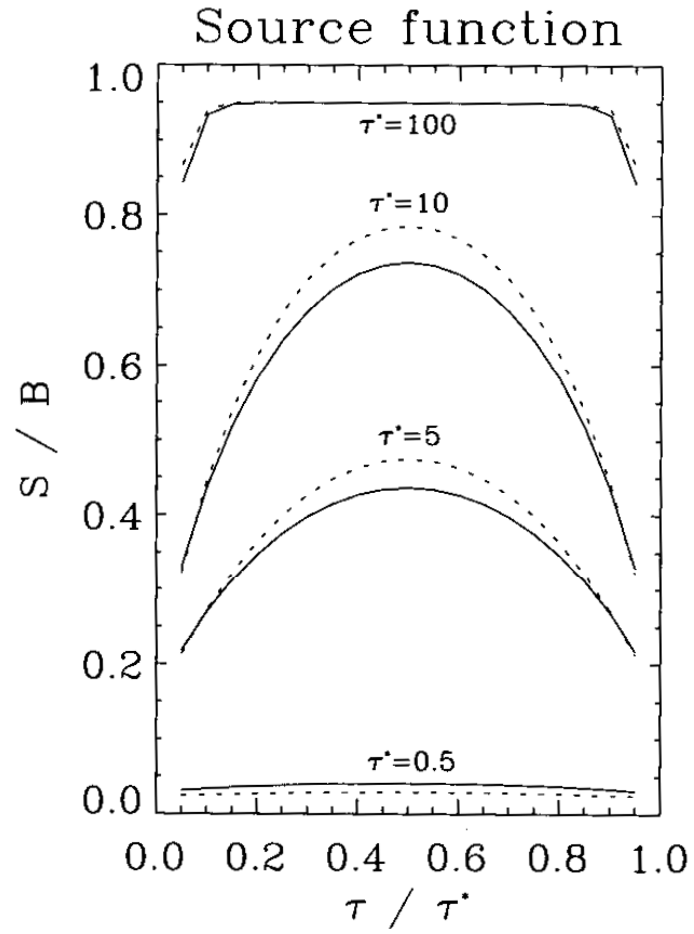
Mean intensity = dashed line

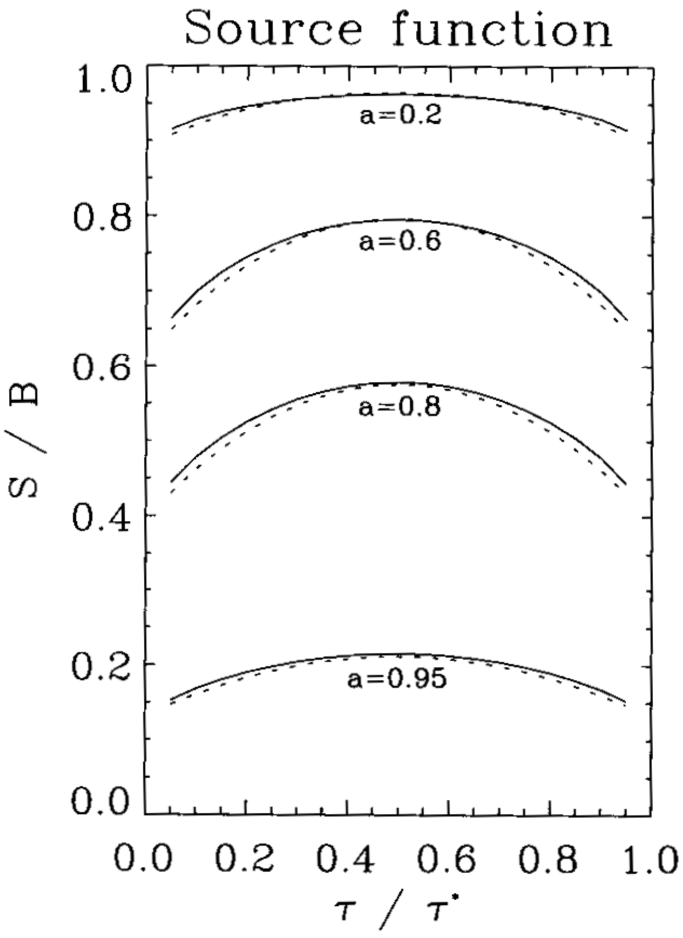


Parameters as above except a=1.0



Source function, S, divided by B





Prototype problem 3

- Assume an isotropically scattering homogeneous atmosphere with a black lower boundary. The appropriate two-stream equations are

$$\bar{\mu}\frac{dI_d^+}{d\tau} = I_d^+ - \frac{a}{2}I_d^+ - \frac{a}{2}I_d^- - \frac{a}{4\pi}F^S e^{-\tau/\mu_0}$$

$$-\bar{\mu}\frac{dI_d^-}{d\tau} = I_d^- - \frac{a}{2}I_d^+ - \frac{a}{2}I_d^- - \frac{a}{4\pi}F^S e^{-\tau/\mu_0}$$

Prototype problem 3

- As before we differentiate and substitute into the equations and get two simultaneous equations

$$\bar{\mu}^2 \frac{d^2(I_d^+ + I_d^-)}{d\tau^2} = (1-a)(I_d^+ + I_d^-) - \frac{a}{4\pi} F^S e^{-\tau/\mu_0}$$

$$\bar{\mu}^2 \frac{d^2(I_d^+ - I_d^-)}{d\tau^2} = (1-a)(I_d^+ - I_d^-) - \frac{a}{4\pi} F^S e^{-\tau/\mu_0}$$

Prototype problem 3

- Using the same solution method as for problem 2 we consider the homogeneous solution

$$I_d^+ = Ae^{\Gamma\tau} + \rho_\infty De^{-\Gamma\tau}, \quad I_d^- = \rho_\infty Ae^{\Gamma\tau} + De^{-\Gamma\tau}$$

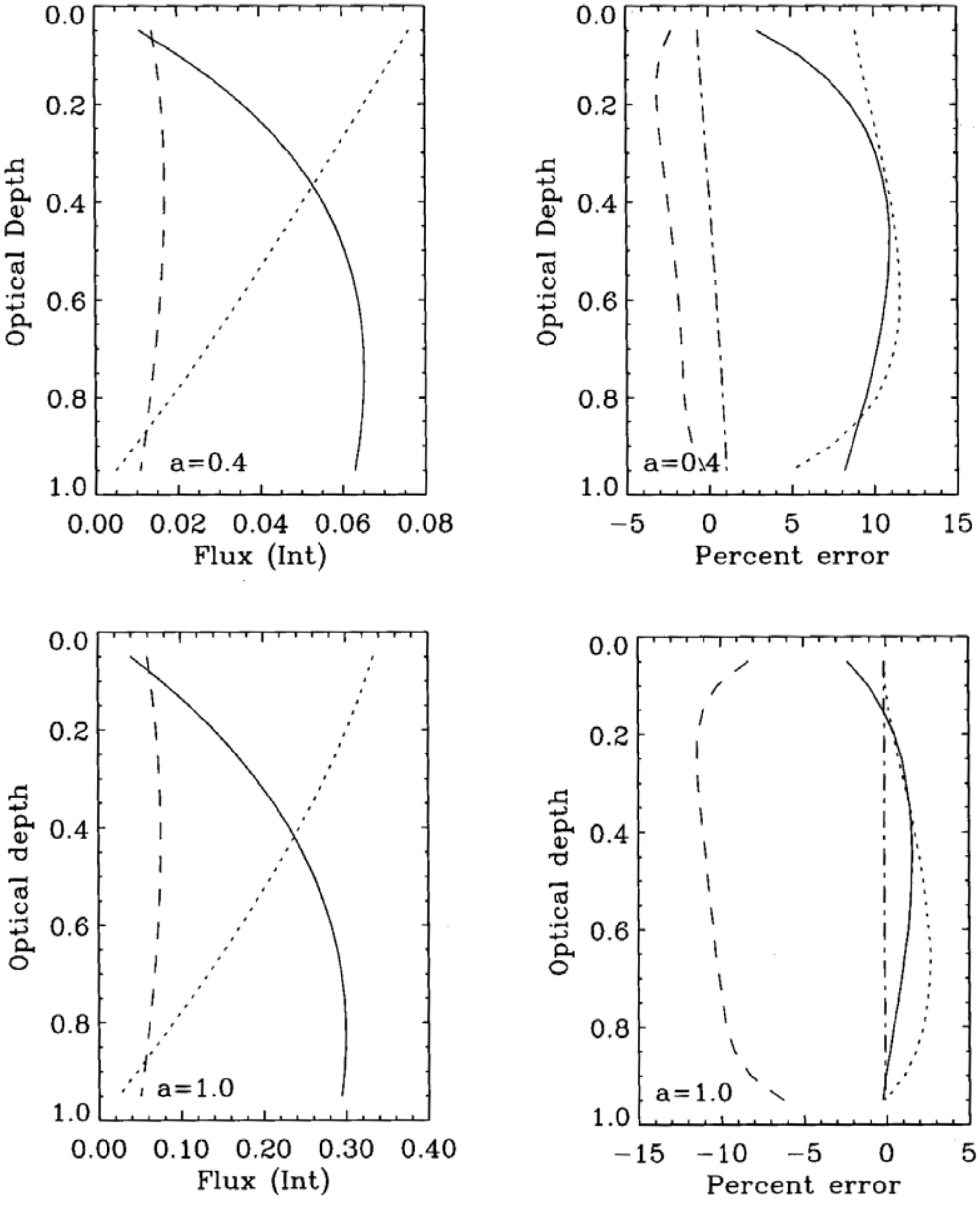
- We now guess that a particular solution will be proportional to exp(-τ/μ_0). We get

$$I_d^+ = Ae^{\Gamma\tau} + \rho_\infty De^{-\Gamma\tau} + \mathrm{Z}^+ e^{-\tau/\mu_0}$$

$$I_d^- = \rho_\infty Ae^{\Gamma\tau} + De^{-\Gamma\tau} + \mathrm{Z}^- e^{-\tau/\mu_0}$$

- Z^+ and Z^- can be determined by substitution

Prototype problem 3



Semi-infinite slab

Essentially we consider the limit $\tau^* \to \infty$. This is an approximation to a very thick planetary atmosphere such as on Venus or Jupiter. Invoking the condition that:

$$S(\tau)e^{-\tau} \to 0$$

then we must exclude the positive exponential terms.

$I^+(\tau) = Ae^{\Gamma\tau} + De^{-\Gamma\tau}$ and $I^-(\tau) = \rho_\infty Ae^{\Gamma\tau} + De^{-\Gamma\tau}$

and from the boundary condition at $\tau = 0$, $D = \bar{I}$

Semi-infinite slab

These equations now become:

$$I^-(\tau) = \bar{I}\, e^{-\Gamma\tau}, \quad I^+(\tau) = \bar{I}\, \rho_\infty e^{-\Gamma\tau}, S(\tau) = \frac{a}{2}\bar{I}(1+\rho_\infty)e^{-\Gamma\tau}$$

$$F(\tau) = -2\pi\, \bar{\mu}\, \bar{I}(1-\rho_\infty)e^{-\Gamma\tau}$$

Note that the flux has a negative sign, indicating that the net flux is downward. It should also be noted that the flux approaches zero as τ becomes large

Eddington Approximation

- Two stream approximations are used primarily to compute fluxes and mean intensities in plane geometry. These quantities depend only on the azimuthally averaged radiation field. We are interested in solutions valid for anisotropic scattering

$$u\frac{dI_d(\tau,u)}{d\tau} = I_d(\tau,u) - \frac{a}{2}\int_{-1}^{1} du'\, p(u',u) I_d(\tau,u') - S^*(\tau,u)$$

Eddington Approximation

- Another approach is to approximate the angular dependence of the intensity by a polynomial in *u*.
- We choose $I(\tau,u)=I_0(\tau)+uI_1(\tau)$
- This approach is referred to as the Eddington approximation. Upon substitution we get

$$u\frac{d(I_0+uI_1)}{d\tau}=I_0+uI_1-\frac{a}{2}\int_{-1}^{1}du'\,p(u',u)(I_0+u'I_1)$$

$$-\frac{aF^S}{4\pi}p(-\mu_0,u)e^{-\tau/\mu_0}$$

Eddington Approximation

- Remember that the phase function can be expanded in terms of Legendre polynomials

$$p(u',u) = \sum_{l=0}^{\infty} (2l-1)\chi_l P_l(u) P_l(u')$$

where the moments χ are given by

$$\chi_l = \frac{1}{2}\int_{-1}^{1} du'\, p(u',u) P_l(u')\, P_l(u)$$

for $l = 0 \;\; \chi_l = 1$, for $l = 1 \;\; \chi_l = g$

Eddington Approximation

- If we only retain these first two terms then the equation becomes

$$\frac{a}{2}\int_{-1}^{1} du'\, p(u',u)(I_0 + u' I_1) = a(I_0 + 3gu\langle u\rangle_2 I_1)$$

$$\text{where}\ \langle u\rangle_2 = \frac{1}{2}\int_{-1}^{1} du u^2$$