AOSC 621
Band Transmittance & Absorptance Model
Transmittance

• For monochromatic radiation the transmittance, $T$, is given simply by

\[ T(\tau; \mu) = e^{-\tau/\mu} \]

and

\[ \frac{dT(\tau; \mu)}{d\tau} = -\frac{1}{\mu} e^{-\tau/\mu} \]

• For finite spectral band, the mean $T$, is given simply by

\[ T_{\Delta \nu} = 1 - \langle \alpha_{\Delta \nu} \rangle = \frac{1}{\Delta \nu} \left( \int_{\Delta \nu} e^{-k,\nu} d\nu \right) \]
Absorptance in spectrally complex media

• So far we have defined the transmittance, $T(\nu)$. We can also define a beam absorptance $\alpha_b(\nu)$, where $\alpha_b(\nu) = 1 - T(\nu)$.

• But in radiative transfer we must consider transmission over a broad spectral range, $\Delta \nu$.

• In this case the mean band absorptance is defined as

$$\langle \alpha_b \rangle = \frac{1}{\Delta \nu} \int_{\Delta \nu} (1 - e^{-k\nu u}) d\nu$$

• Here $u$ is the total mass along the path, and $k$ is the monochromatic mass absorption coefficient.
Absorption in a Lorentz line shape
Transmission in spectrally complex media

- Consider a single line with a Lorentz line shape.
- As \( u \) gets larger the absorptance gets larger. However as \( u \) increases the center of the line absorbs all of the radiation, while the wings absorb only some of the radiation.
- At this point only the wings absorb.
- We can write the band transmittance as:

\[
T_{\Delta \nu} = \frac{1}{\Delta \nu} \int_{\Delta \nu} \exp \left( - \frac{S \alpha_L u}{\pi((\nu - \nu_0) + \alpha^2_L)} \right) d\nu
\]
Non-overlapping lines

The mean spectral beam transmittance and absorptance of an isolated line is:

\[
\langle T_b \rangle = \frac{1}{\Delta \phi} \int \exp[-\alpha_m(\phi)] d\phi
\]

\[
\langle \alpha_b \rangle = \frac{1}{\Delta \phi} \int [1 - \exp[-\alpha_m(\phi)]] d\phi
\]

If \( n \) lines within \( \Delta \phi \) contribute to the absorption at a particular wavenumber:

\[
\alpha_m(\phi) = \sum_{j=1}^{n} \alpha_m^j(\phi)
\]

If there is no overlap, total absorptance is

\[
W = \sum_{j=1}^{n} W_j = \sum_{j=1}^{n} \int \left[1 - \exp[-\alpha_m^j(\phi)]\right] d\phi
\]
Influence of overlap

Consider two *identical* lines within $\Delta \sqrt{\nu}$. When completely separated:

$$W = 2 \int_{\Delta \sqrt{\nu}} [1 - \exp(-\alpha_m(\sqrt{\nu}))] d\sqrt{\nu}$$

When perfectly superimposed:

$$W' = \int_{\Delta \sqrt{\nu}} [1 - \exp(-2\alpha_m(\sqrt{\nu}))] d\sqrt{\nu}$$

If several lines contribute to the transmittance (absorptance) at a particular wavenumber $\tilde{\sqrt{\nu}}$, the total transmittance is the product of transmittances:

$$T_b(\tilde{\sqrt{\nu}}) = \exp[-\sum_{j=1}^{n} \alpha_m(\sqrt{\nu}_j)] = \prod_{j=1}^{n} \exp[-\alpha_m(\sqrt{\nu}_j)] = \prod_{j=1}^{n} T_b^j$$
Line-By-Line (LBL) models

Monochromatic optical path along inhomogeneous path:

\[ \tau(\varphi) = \int_{u} \alpha_m(\varphi u')du' = \int_{u} \sum_{j=1}^{n} \alpha_j^m(\varphi p, T)du' = \int_{u} \sum_{j=1}^{n} S_j(\varphi T)\Phi(\varphi p, T)du' \]

Need to resolve individual lines!

The average spectral beam transmittance is:

\[ \approx \frac{1}{\Delta \varphi} \sum_{i=1}^{M} \exp \left( -\sum_{i=1}^{L} \sum_{j=1}^{n} \alpha_j^i(\varphi_i p_i, T_i)\Delta u_i' \right) \Delta \varphi \]

when broken to smaller paths that can be considered homogeneous

Lots of calculations!
Transmission in spectrally complex media

• Let $\Delta \nu$ be large enough to include all of the line, then we can extend the limits of the integration to $\pm\infty$.

• Landenburg and Reichle showed that

$$T_{\Delta \nu} = 1 - \frac{1}{\Delta \nu} e^{-x} \left[ J_0(ix) - iJ_1(ix) \right]$$

where $x = \frac{Su}{2\pi \alpha_L}$, $J_0$ and $J_1$ are Bessel functions
Transmission in spectrally complex media

- For small $x$, known as the optically thin condition,

$$T = 1 - \frac{Su}{\Delta \nu}$$

for large $x$, optically thick,

$$T = 1 - \frac{2Su}{\Delta \nu \sqrt{2\pi x}} = 1 - \frac{2}{\Delta \nu} \sqrt{\alpha \nu Su}$$
Elsasser band model
• Elsasser and Culbertson (1960) assumed evenly spaced lines \((d\) apart) of equal \(S\) that overlapped. They showed that the transmittance could be expressed as

\[
T = \int_{-\infty}^{\infty} \exp[-y\coth(\beta)] J_0(iy\sinh(\beta)) dy
\]

where \(\beta = \frac{2\pi\alpha_L}{d}\) and \(y = \frac{Su}{d\sinh(\beta)}\)

Optically thin

\[
T = 1 - \frac{Su}{d}
\]

Optically thick

\[
T = 1 - \frac{2}{d} \sqrt{\alpha_L Su}
\]
Statistical Band Model (Goody)

• Goody studied the water-vapor bands and noted the apparent random line positions and band strengths.
• Let us assume that the interval $\Delta \nu$ contains $n$ lines of mean separation $d$, i.e. $\Delta \nu = nd$
• Let the probability that line $i$ has a line strength $S_i$ be $P(S_i)$ where

$$\int_{0}^{\infty} P(S_i) dS = 1$$

• $P$ is normally assumed to have a Poisson distribution
Statistical Band Model (Goody)

• For any line the most probable value of $S$ is given by

$$\bar{S} = \frac{\int_{0}^{\infty} SP(S) dS}{\int_{0}^{\infty} P(S) dS}$$

• The transmission $T$ over an interval $\Delta\nu$ is given by

$$T_i = \frac{1}{\Delta\nu} \int d\nu \int_{\Delta\nu}^{\infty} P(S_i) e^{-k_i u} dS_i$$
Statistical Band Model (Goody)

- For $n$ lines the total transmission $T$ is given by

$$T = T_1 T_2 T_3 T_4 \cdots \cdots \cdots T_n$$

$$T = \left( \frac{1}{(\Delta \nu)^n} \int_{\Delta \nu} \int_{\Delta \nu} \int_{\Delta \nu} \cdots \int_{\Delta \nu} d\nu_1 \right)^n d\nu_2$$

$$\times \int_{0}^{\infty} P(S_1) e^{-k_{1u}} dS_1 \int_{0}^{\infty} P(S_2) e^{-k_{2u}} dS_2 \cdots \cdots \int_{0}^{\infty} P(S_n) e^{-k_{nu}} dS_1$$

$$T = \left( \frac{1}{(\Delta \nu)^n} \int_{\Delta \nu} d\nu \int_{0}^{\infty} P(S) e^{-k_{u}} dS \right)^n$$

$$A = \left[ 1 - \frac{1}{(\Delta \nu)^n} \int_{\Delta \nu} d\nu \int_{0}^{\infty} P(S) \left( 1 - e^{-k_{u}} \right) dS \right]^n$$
Statistical Band Model (Goody)

The equation can be further simplified to

\[ T \approx \exp \left[ -\frac{1}{d} \int d\nu \int_0^\infty P(S)(1-e^{ku})dS \right] \]

assuming a Lorentz line shape, one gets

\[ T = \exp \left[ -\frac{\bar{S}u}{d} \left( 1 + \frac{\bar{S}u}{\pi\alpha_L} \right)^{-1/2} \right] \]

where \( \bar{S} \) is the mean line strength.
Statistical Band Model (Goody)

• We have reduced the parameters needed to calculate $T$ to two

\[
\frac{\bar{S}}{d} \quad \text{and} \quad \frac{\pi \alpha_L}{d}
\]

• These two parameters are either derived by fitting the values of $T$ obtained from a line-by-line calculation, or from experimental data.

<table>
<thead>
<tr>
<th>weak line</th>
<th>strong line</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T = \exp\left(-\frac{\bar{S}u}{d}\right)$</td>
<td>$T = \exp\left(-\frac{\sqrt{\pi \alpha_L \bar{S}u}}{d}\right)$</td>
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</tbody>
</table>
Summary of band models

<table>
<thead>
<tr>
<th>Band model</th>
<th>Strong approximation</th>
<th>Weak approximation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ladenburg-Reiche</td>
<td>$\tau = 1 - \frac{2}{\Delta v} \sqrt{\alpha S u}$</td>
<td>$\tau = 1 - \frac{S u}{\Delta v}$</td>
</tr>
<tr>
<td>(single Lorentz line)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Elsasser</td>
<td>$\tau = 1 - \text{erf} \frac{\sqrt{\pi \alpha S u}}{d}$</td>
<td>$\tau = 1 - \frac{S u}{d}$</td>
</tr>
<tr>
<td>Goody random</td>
<td>$\tau = \exp \left[ - \frac{\sqrt{\pi \alpha S_0 u}}{d} \right]$</td>
<td>$\tau = \exp \left[ - \frac{S_0 \mu}{d} \right]$</td>
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</tbody>
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K-Distribution Method: the concept

Choices for calculating spectral average beam transmittances:

1. Line-by-Line (LBL): accurate, but lots of CPU
2. Band models: fast, but not so accurate
3. k-distribution: fast, with acceptable accuracy

The basic idea behind the k-distribution

\[
\langle T_b(u) \rangle = \frac{1}{\Delta \lambda} \int_{\lambda_1}^{\lambda_2} e^{-k(\theta) u} d\lambda \approx \frac{1}{\lambda_2 - \lambda_1} \int_{\lambda_1}^{\lambda_2} e^{-k(\theta) u} d\lambda \approx \sum_{l=1}^{L} e^{-k_l u} \frac{\Delta \lambda}{\Delta \lambda}
\]

Same (approximate) value of \( \kappa^2 \) is encountered many times!
The concept (continued)

The average beam spectral transmittance can be written

\[ \langle T_b(u) \rangle \approx \sum_{j=1}^{N} f(k_j) e^{-k_j u} \Delta k_j \]

Note

\[ \int_{k_{\min}}^{k_{\max}} f(k) dk = 1 \]
Cumulative distributions and integral forms

Now, let’s define a cumulative distribution \( \gamma(\kappa) \) such that:

\[
g(k_m) = \sum_{j=1}^{m} f(k_j) \Delta k_j \Rightarrow g(k_N) = g(k_{\text{max}}) = \sum_{j=1}^{N} f(k_j) \Delta k_j = 1
\]

\[
\langle T_b(u) \rangle \approx \sum_{j=1}^{N} f(k_j) e^{-k_j u} \Delta k_j = \sum_{j=1}^{N} \Delta g(k_j) e^{-k_j u} \equiv \sum_{j=1}^{N} \Delta g_j e^{-k_j u}
\]

\[
\langle T_b(u) \rangle = \int_{0}^{1} d g e^{-k(g) u} \quad g(k) = \int_{0}^{k} f(k') dk'
\]
Example 2

9.6 μm band of O₃ at 30 mb