AOSC 621
Band Transmittance & Absorptance Model

Transmittance

- For monochromatic radiation the transmittance, *T*, is given simply by

$$T(\tau;\mu) = e^{-\tau/\mu}$$

and

$$\frac{dT(\tau;\mu)}{d\tau} = -\frac{1}{\mu} e^{-\tau/\mu}$$

- For finite spectral band, the mean *T*, is given simply by

$$T_{\Delta\nu} = 1 - \langle \alpha_{\Delta\nu} \rangle = \frac{1}{\Delta\nu}\left(\int_{\Delta\nu} e^{-k_\nu u} d\nu \right)$$

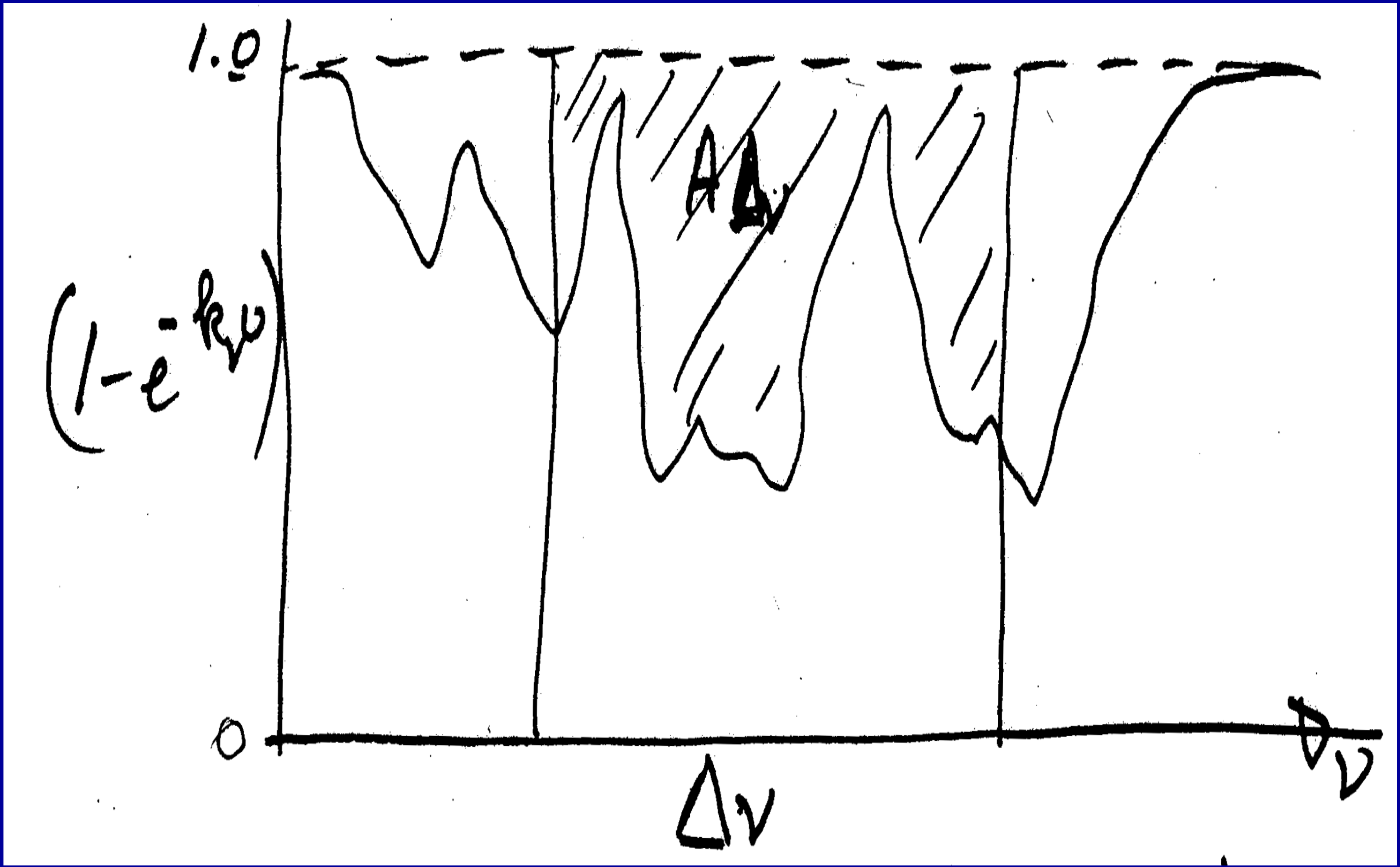
Absorptance in spectrally complex media

- So far we have defined the transmittance, $T(\nu)$. We can also define a beam absorptance $\alpha_b(\nu)$, where $\alpha_b(\nu) = 1 - T(\nu)$.
- But in radiative transfer we must consider transmission over a broad spectral range, $\Delta\nu$.
- In this case the mean band absorptance is defined as

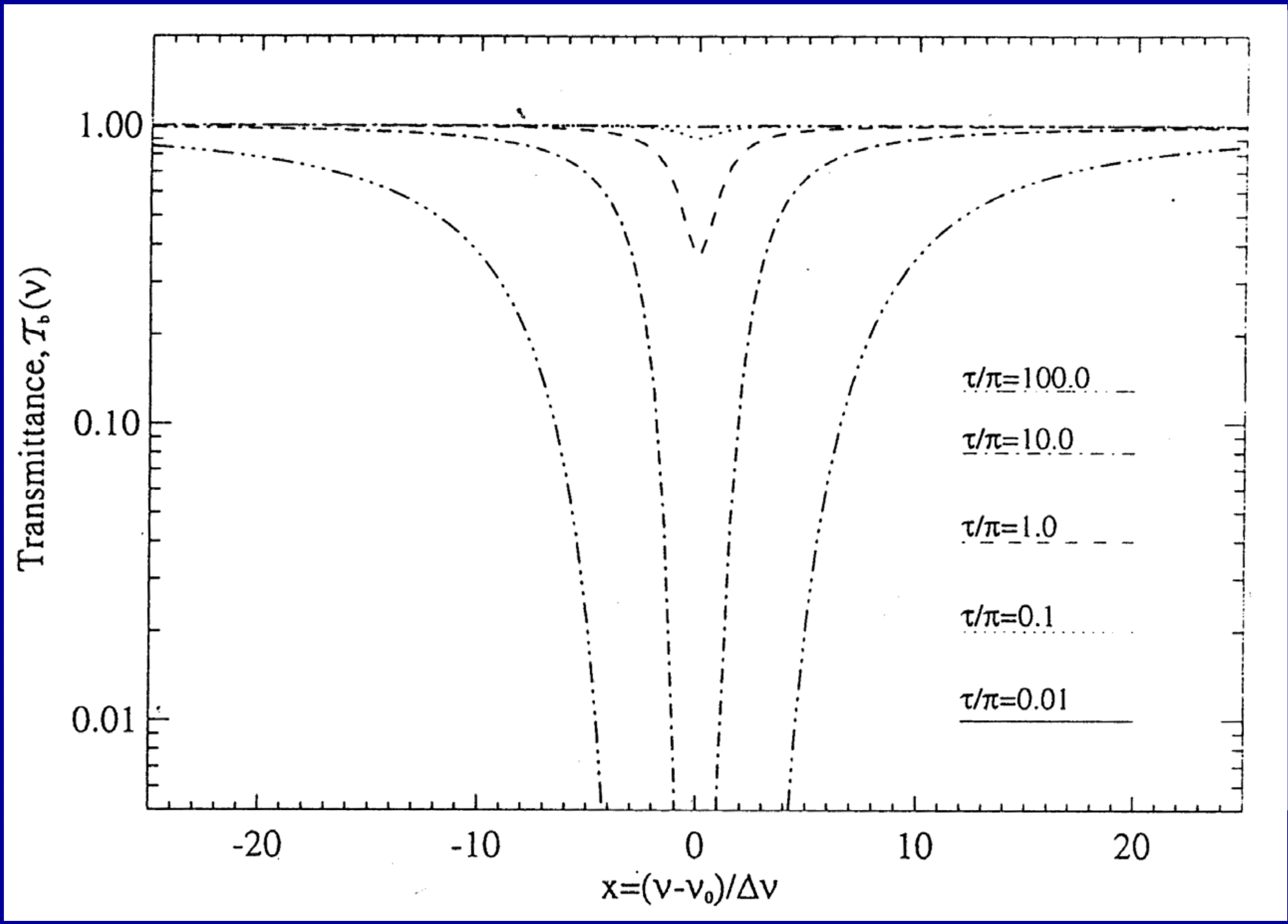
$$\langle \alpha_b \rangle = \frac{1}{\Delta \nu} \int_{\Delta \nu} (1 - e^{-k_\nu u}) d\nu$$

- Here u is the total mass along the path, and k is the monochromatic mass absorption coefficient.

Schematic for A



Absorption in a Lorentz line shape



Transmission in spectrally complex media

- Consider a single line with a Lorentz line shape.
- As u gets larger the absorptance gets larger. However as u increases the center of the line absorbs all of the radiation, while the wings absorb only some of the radiation.
- At this point only the wings absorb.
- We can write the band transmittance as;

$$T_{\Delta\nu} = \frac{1}{\Delta\nu}\int_{\Delta\nu} \exp\left(-\left[\frac{S\alpha_L u}{\pi((\nu-\nu_0)+\alpha_L^2)}\right]\right) d\nu$$

Non-overlapping lines

The mean spectral beam transmittance and absorptance of an *isolated* line is:

$$\langle T_b \rangle = \frac{1}{\Delta \tilde{\nu}} \int_{\Delta \tilde{\nu}} \exp[-\alpha_m(\tilde{\nu}) u] \, d\tilde{\nu}$$

$$\langle \alpha_b \rangle = \frac{1}{\Delta \tilde{\nu}} \int_{\Delta \tilde{\nu}} \left[1 - \exp[-\alpha_m(\tilde{\nu}) u] \right] d\tilde{\nu}$$

If n lines within $\Delta \tilde{\nu}$ contribute to the absorption at a particular wavenumber:

$$\alpha_m(\tilde{\nu}) = \sum_{j=1}^{n} \alpha_m^j(\tilde{\nu})$$

If there is no overlap, total absorptance is

$$W = \sum_{j=1}^{n} W_j = \sum_{j=1}^{n} \int_{\frac{\Delta \tilde{\nu}}{n}} \left[1 - \exp[-\alpha_m^j(\tilde{\nu}) u] \right] d\tilde{\nu}$$

Influence of overlap

Consider two *identical* lines within $\Delta\tilde{\nu}$. When completely separated:

$$W = 2\int_{\Delta\tilde{\nu}} \left[1 - \exp[-\alpha_m(\tilde{\nu})u]\right] d\tilde{\nu}$$

When perfectly superimposed:

$$W' = \int_{\Delta\tilde{\nu}} \left[1 - \exp[-2\alpha_m(\tilde{\nu})u]\right] d\tilde{\nu}$$

If several lines contribute to the transmittance (absorptance) at a particular wavenumber $\tilde{\nu}$, the total transmittance is the product of transmittances:

$$T_b(\tilde{\nu}) = \exp[-\sum_{j=1}^{n} \alpha_m^j(\tilde{\nu})u_j] = \prod_{j=1}^{n} \exp[-\alpha_m^j(\tilde{\nu})u_j] = \prod_{j=1}^{n} T_b^j$$

Line-By-Line (LBL) models

Monochromatic optical path along inhomogeneous path:

$$\tau(\tilde{\nu}) = \int_u \alpha_m(\tilde{\nu},u')du' = \int_u \sum_{j=1}^{n} \alpha_m^j(\tilde{\nu},p,T)du' = \int_u \sum_{j=1}^{n} S_j(\tilde{\nu},T)\Phi(\tilde{\nu},p,T)du'$$

Need to resolve individual lines!

The average spectral beam transmittance is:

$$\approx \frac{1}{\Delta\tilde{\nu}} \sum_{i=1}^{M} \exp\left(-\sum_{l=1}^{L}\sum_{j=1}^{n} \alpha_m^j(\tilde{\nu}_i, p_l, T_l)\Delta u_l'\right)\Delta\tilde{\nu}_i$$

when broken to smaller paths that can be considered homogeneous

<u>Lots of calculations!</u>

Transmission in spectrally complex media

- Let $\Delta\nu$ be large enough to include all of the line, then we can extend the limits of the integration to $\pm\infty$.
- Landenburg and Reichle showed that

$$T_{\Delta\nu} = 1 - \frac{1}{\Delta\nu} e^{-x}\left[J_0(ix) - iJ_1(ix)\right]$$

where $x = \dfrac{Su}{2\pi\alpha_L}$ $\qquad J_0$ and J_1 are Bessel functions

Transmission in spectrally complex media

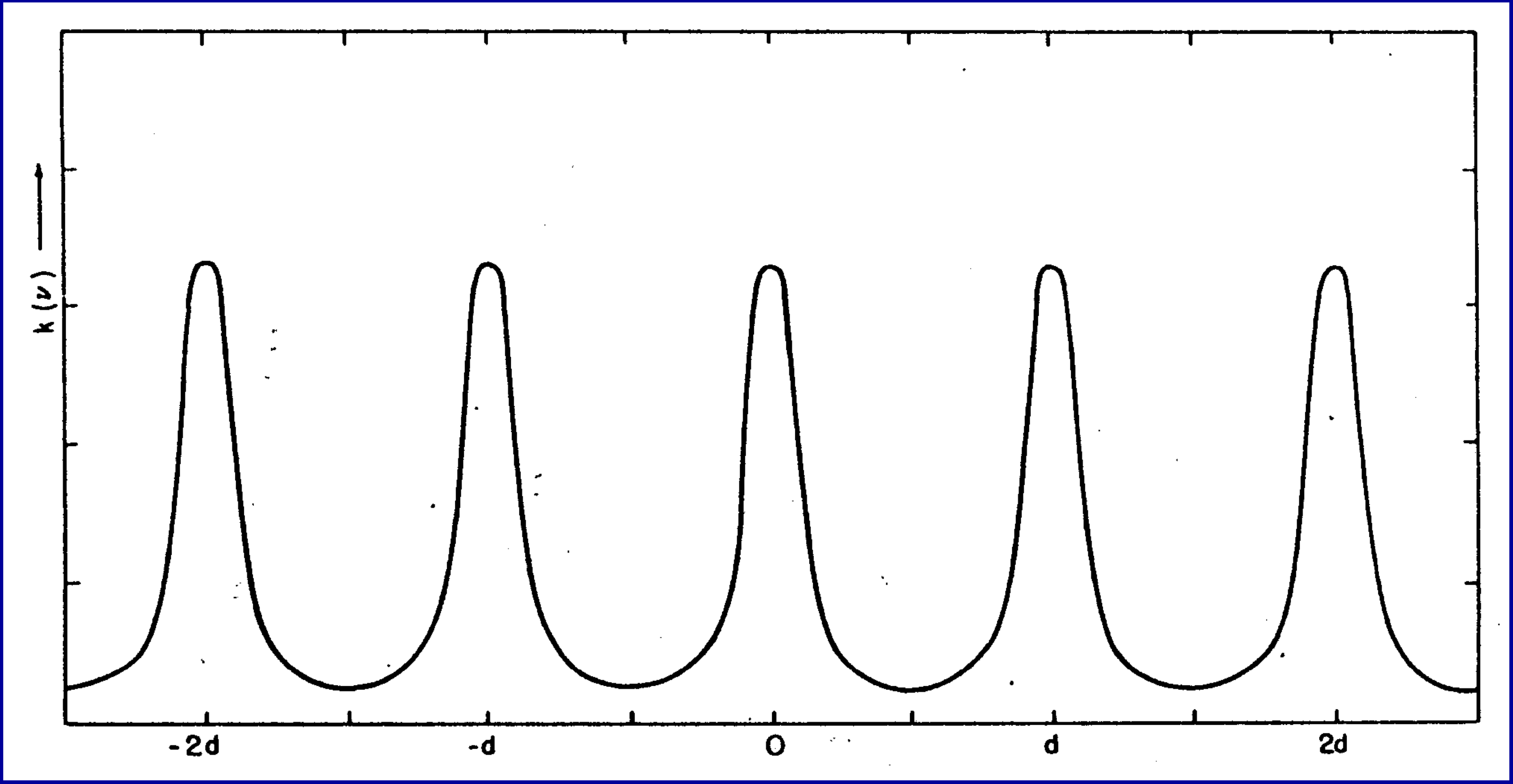
- For small x , known as the optically thin condition,

$$T = 1 - \frac{Su}{\Delta\nu}$$

for large x, optically thick,

$$T = 1 - \frac{2Su}{\Delta\nu\sqrt{2\pi x}} = 1 - \frac{2}{\Delta\nu}\sqrt{\alpha_L Su}$$

Elsasser band model



Elsasser Band Model

- Elsasser and Culbertson (1960) assumed evenly spaced lines (*d* apart) of equal *S* that overlapped. They showed that the transmittance could be expressed as

$$T = \int_{-\infty}^{\infty} \exp\left[-y\coth(\beta)\right] J_0(iy\sinh(\beta))\,dy$$

$$\text{where } \beta = \frac{2\pi\alpha_L}{d} \quad \text{and} \quad y = \frac{Su}{d\sinh(\beta)}$$

optically thin

$$T = 1 - \frac{\mathrm{Su}}{\mathrm{d}}$$

optically thick

$$T = 1 - \frac{2}{\mathrm{d}}\sqrt{\alpha_L S u}$$

Statistical Band Model (Goody)

- Goody studied the water-vapor bands and noted the apparent random line positions and band strengths.
- Let us assume that the interval $\Delta\nu$ contains n lines of mean separation d, i.e. $\Delta\nu = nd$
- Let the probability that line i has a line strength S_i be $P(S_i)$ where

$$\int_0^\infty P(S_i)\,dS = 1$$

- P is normally assumed to have a Poisson distribution

Statistical Band Model (Goody)

- For any line the most probable value of S is given by

$$\bar{S} = \frac{\int_0^\infty S P(S)\, dS}{\int_0^\infty P(S)\, dS}$$

- The transmission T over an interval Δν is given by

$$T_i = \frac{1}{\Delta\nu} \int_{\Delta\nu} d\nu \int_0^\infty P(S_i) e^{-k_i u} dS_i$$

Statistical Band Model (Goody)

- For n lines the total transmission T is given by

  $T = T_1 T_2 T_3 T_4 \ldots\ldots\ldots\ldots T_n$

$$T = \frac{1}{(\Delta\nu)^n} \int_{\Delta\nu} d\nu_1 \int_{\Delta\nu} d\nu_1 \ldots\ldots\ldots\ldots\ldots\ldots \int_{\Delta\nu} d\nu_n$$

$$\text{x} \int_0^\infty P(S_1) e^{-k_1 u} dS_1 \int_0^\infty P(S_2) e^{-k_2 u} dS_2 \ldots\ldots\ldots \int_0^\infty P(S_n) e^{-k_n u} dS_1$$

$$T = \frac{1}{(\Delta\nu)^n} \left[\int_{\Delta\nu} d\nu \int_0^\infty P(S) e^{-ku} dS \right]^n$$

$$A = \left[1 - \frac{1}{\Delta\nu} \int_{\Delta\nu} d\nu \int_0^\infty P(S)(1 - e^{-ku}) dS \right]^n$$

Statistical Band Model (Goody)

The equation can be further simplified to

$$T \cong \exp\left[-\frac{1}{d}\int_{\Delta\nu} d\nu \int_{o}^{\infty} P(S)(1-e^{ku})dS\right]$$

assuming a Lorentz line shape, one gets

$$T = \exp\left[-\frac{\bar{S}u}{d}\left(1+\frac{\bar{S}u}{\pi\alpha_L}\right)^{-1/2}\right]$$

where $\bar{S}$ is the mean line strength.

Statistical Band Model (Goody)

- We have reduced the parameters needed to calculate T to two

$$\frac{\bar{S}}{d} \quad \text{and} \quad \frac{\pi\alpha_L}{d}$$

- These two parameters are either derived by fitting the values of T obtained from a line-by-line calculation, or from experimental data.

weak line

$$T = \exp\left(-\frac{\bar{S}u}{d}\right)$$

strong line

$$T = \exp\left(-\frac{\sqrt{\pi\alpha_L \bar{S}u}}{d}\right)$$

Summary of band models

Summary of the strong and weak approximations of the three band models

Band model	Strong approximation	Weak approximation
Ladenburg-Reiche (single Lorentz line)	$\tau = 1 - \frac{2}{\Delta\nu}\sqrt{\alpha S u}$	$\tau = 1 - \frac{Su}{\Delta\nu}$
Elsasser	$\tau = 1 - \mathrm{erf}\,\frac{\sqrt{\pi\alpha S u}}{d}$	$\tau = 1 - \frac{Su}{d}$
Goody random	$\tau = \exp\left[-\frac{\sqrt{\pi\alpha S_0 u}}{d}\right]$	$\tau = \exp\left[-\frac{S_0 u}{d}\right]$

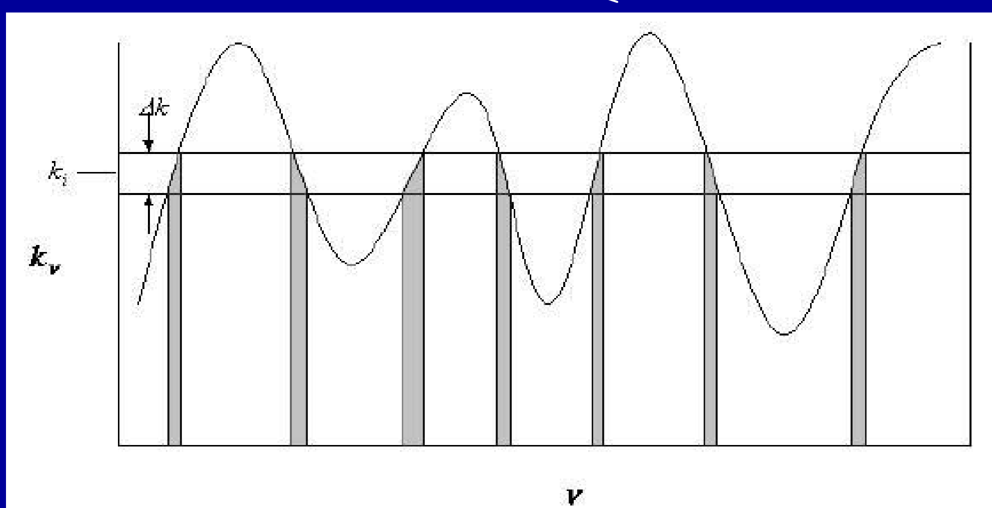
K-Distribution Method: the concept

Choices for calculating spectral average beam transmittances:

1. Line-by-Line (LBL): accurate, but lots of CPU
2. Band models: fast, but not so accurate
3. k-distribution: fast, with acceptable accuracy

The basic idea behind the k-distribution

$$\langle T_b(u)\rangle = \frac{1}{\Delta\nu}\int_{\Delta\nu} e^{-k(\nu)u}\,d\nu = \frac{1}{\nu_2-\nu_1}\int_{\nu_1}^{\nu_2} e^{-k(\nu)u}\,d\nu \approx \sum_{l=1}^{L} e^{-k^l u}\,\frac{\Delta\nu_l}{\Delta\nu}$$



Same (approximate) value of κ^λ is encountered many times!

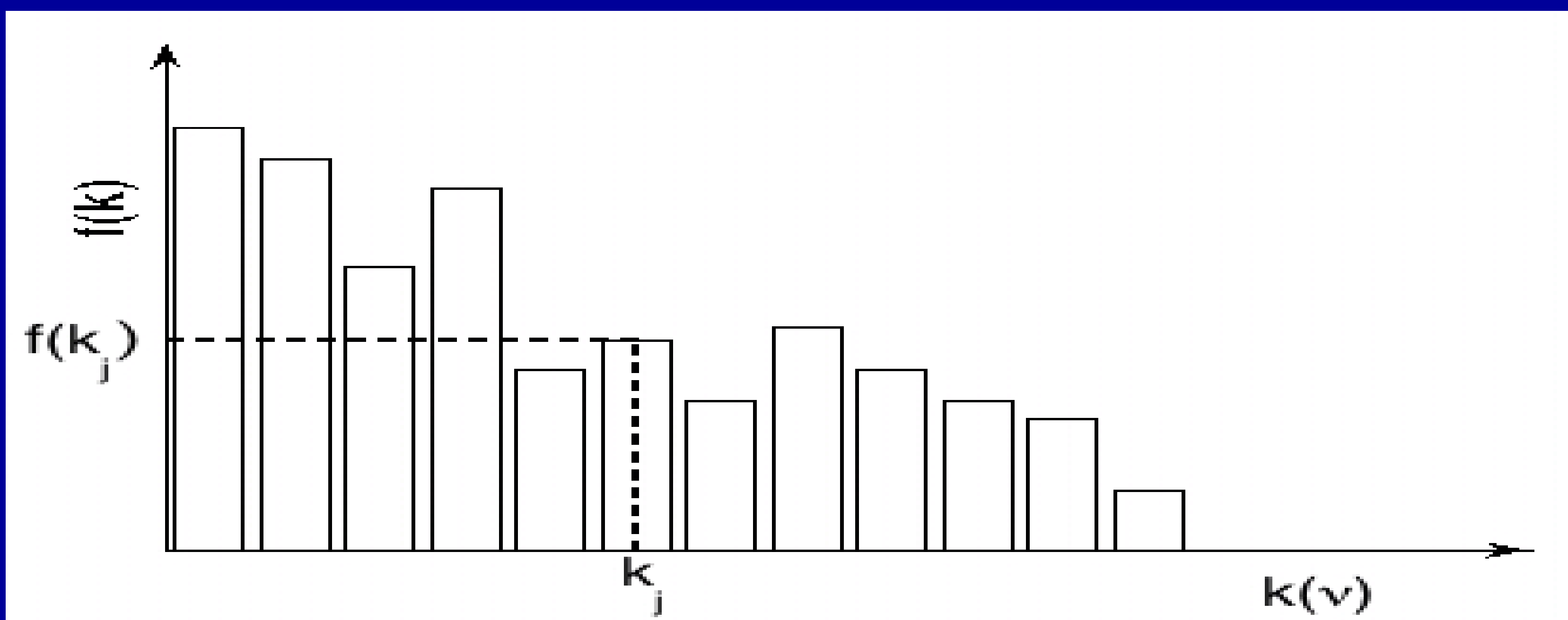
The concept (continued)

The average beam spectral transmittance can be written

$$\langle T_b(u) \rangle \approx \sum_{j=1}^{N} f(k_j) e^{-k_j u} \Delta k_j$$

Note

$$\int_{k_{\min}}^{k_{\max}} f(k)\,dk = 1$$



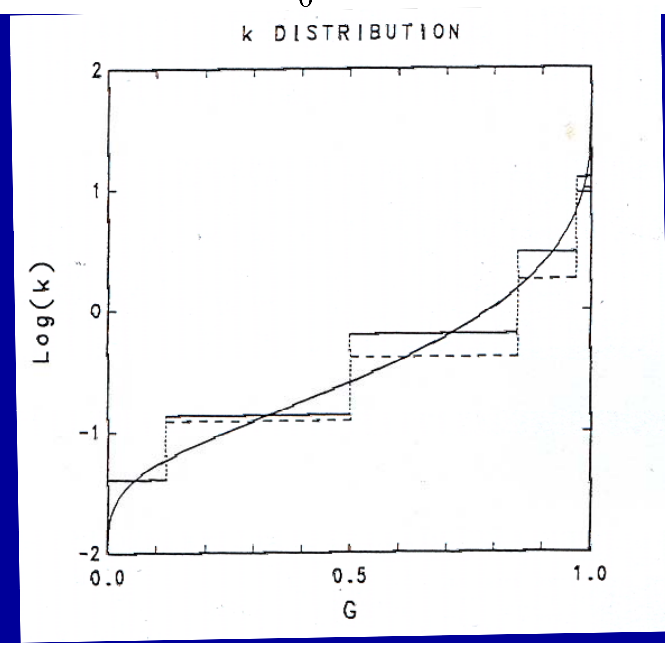
Cumulative distributions and integral forms

Now, let's define a cumulative distribution $\gamma(\kappa)$ such that:

$$g(k_m) \equiv \sum_{j=1}^{m} f(k_j)\Delta k_j \Rightarrow g(k_N) = g(k_{\max}) = \sum_{j=1}^{N} f(k_j)\Delta k_j = 1$$

$$\langle T_b(u) \rangle \approx \sum_{j=1}^{N} f(k_j)e^{-k_j u}\Delta k_j = \sum_{j=1}^{N} \Delta g(k_j)e^{-k_j u} \equiv \sum_{j=1}^{N} \Delta g_j e^{-k_j u}$$

$$\langle T_b(u) \rangle = \int_0^1 dg e^{-k(g)u} \qquad g(k) = \int_0^k f(k')dk'$$



Example 2

9.6 µm band of O_3 at 30 mb