METO 621

Lesson 9

Half Range Quantities

- In the limit of no scattering the radiative transfer equations for the half-range intensities become

$$\mu \frac{dI_\nu^+(\tau,\mu,\phi)}{d\tau} = I_\nu^+(\tau,\mu,\phi) - B(\tau)$$

$$-\mu \frac{dI_\nu^-(\tau,\mu,\phi)}{d\tau} = I_\nu^-(\tau,\mu,\phi) - B(\tau)$$

Upper half-range intensity

- For the upper half-range intensity we use the integrating factor $e^{-\tau/\mu}$

$$\frac{d}{d\tau}\left(I_\nu^+ e^{-\tau/\mu}\right)=\left(\frac{dI_\nu^+}{d\tau}-\frac{1}{\mu}I_\nu^+\right)e^{-\tau/\mu}=-\frac{B_\nu(\tau)}{\mu}e^{-\tau/\mu}$$

- In this case we are dealing with up going beams and we integrate from the bottom to the top.

Upper half-range intensity

$$\int_{\tau^*}^{0} d\tau' \frac{d}{d\tau'}\left(I_\nu^+ e^{-\tau'/\mu}\right) = I_\nu^+(0,\mu,\phi) - I_\nu^+(\tau^*,\mu,\phi)e^{-\tau^*/\mu}$$

$$= -\int_{\tau^*}^{0} \frac{d\tau'}{\mu} e^{-\tau'/\mu} B_\tau(\tau') = \int_{0}^{\tau^*} \frac{d\tau'}{\mu} e^{-\tau'/\mu} B_\tau(\tau')$$

or

$$I_\nu^+(0,\mu,\phi) = I_\nu^+(\tau^*,\mu,\phi)e^{-\tau^*/\mu} + \int_{0}^{\tau^*} \frac{d\tau'}{\mu} B_\nu(\tau')e^{-\tau'/\mu}$$

Upper half-range intensity

- To find the intensity at an interior point τ, integrate from τ^* to τ and obtain

$$I_\nu^+(\tau,\mu,\phi) = I_\nu^+(\tau^*,\mu,\phi)e^{-(\tau^*-\tau)/\mu} + \int_\tau^{\tau^*} \frac{d\tau'}{\mu} B_\nu(\tau')e^{-(\tau'-\tau)/\mu}$$

- What happens when μ approaches zero. This is where the line of sight traverses an infinite distance parallel to the slab. Here

$$I_\tau^\pm(\tau,\mu=0,\phi) = B_\nu(\tau)$$

Formal solution including Scattering and Emission

- Note that the source is now due to thermal emission and multiple scattering

$$S(\tau,\hat{\Omega}) = [1 - a(\tau)]B(\tau) + \frac{a(\tau)}{4\pi}\int_{4\pi} d\omega' \, p(\tau,\hat{\Omega}',\hat{\Omega}) I(\tau,\hat{\Omega}')$$

- The independent variable is the extinction optical depth, the sum of the absorption and scattering optical depths. We can write

$$\mu \frac{dI(\tau,\hat{\Omega})}{d\tau} = -I(\tau,\hat{\Omega}) + S(\tau,\hat{\Omega})$$

Formal solution including scattering and emission

- The method of using an integrating factor can be applied as before

$$I\left[\tau(P_2),\hat{\Omega}\right]=I\left[\tau(P_1),\hat{\Omega}\right]e^{-\tau(P1,P2)}+\int_{\tau(P_1)}^{\tau(P_2)} dt\, S(t,\hat{\Omega})e^{-\tau(P\ P_2)}$$

- In slab geometry the solutions become

$$I^-(\tau,\mu,\phi)=I^-(0,\mu,\phi)e^{-\tau/\mu}+\int_0^{\tau}\frac{d\tau'}{\mu}S(\tau',\mu,\phi)e^{-(\tau-\tau')/\mu}$$

Formal solution including scattering and emission

$$I^{+}(\tau,\mu,\phi)=I^{+}(\tau^{*},\mu,\phi)e^{-(\tau^{*}-\tau)/\mu}+\int_{\tau}^{\tau^{*}}\frac{d\tau'}{\mu}S(\tau')e^{-(\tau'-\tau)/\mu}$$

- where

$$I^{\pm}(\tau,\mu=0,\phi)=S(\tau,\mu=0,\phi)$$

- and

$$S(\tau,\mu,\phi)=[1-a]B(\tau)$$
$$+\frac{a}{4\pi}\int_{0}^{2\pi}d\phi'\int_{0}^{1}d\mu'\,p(\mu',\phi';\mu,\phi)I^{+}(\tau,\mu',\phi')$$
$$+\frac{a}{4\pi}\int_{0}^{2\pi}d\phi'\int_{0}^{1}d\mu'\,p(-\mu',\phi';\mu,\phi)I^{-}(\tau,\mu',\phi')$$

Radiative Heating Rate

- The differential change of energy over the distance *ds* along a beam is

$$\partial(d^4 E) = dI_\nu dA\, dt\, d\nu\, d\omega$$

- If we divide this expression by *dsdA*, (the unit volume, *dV*), and also $d\nu dt$ then we get the time rate of change in radiative energy per unit volume per unit frequency, due to a change in intensity for beams within the solid angle $d\omega$.

Radiative Heating Rate

- Since there is (generally) incoming radiation from all directions, the total change in energy per unit frequency per unit time per unit volume is

$$\int_{4\pi} d\omega \frac{dI_\nu}{ds} = \int_{4\pi} d\omega (\hat{\Omega} \cdot \nabla I_\nu)$$

- The spectral heating rate *H* is

$$H_\nu = -\int_{4\pi} d\omega (\hat{\Omega} \cdot \nabla I_\nu)$$

Radiative Heating Rate

- The net radiative heating rate H is

$$H = -\int_0^{\infty} d\nu \int_{4\pi} d\omega(\hat{\Omega} \cdot \nabla I_{\nu})$$

- In a slab geometry the radiative heating rate is written

$$H = -\int_0^{\infty} d\nu \frac{\delta F_{\nu}}{\delta z} = -2\pi \int_0^{\infty} d\nu \int_{-1}^{+1} \cos\theta d(\cos\theta) \frac{\partial I_{\nu}}{\partial z}$$

where $F_{\nu} = F_{\nu}^{+} - F_{\nu}^{-}$ is the radiative flux in the z direction

Separation into diffuse and direct(Solar) components

- Two distinctly different components of the shortwave radiation field. The solar component:

$$I_S^-(\tau,\theta,\phi) = F^S e^{-\tau/\mu_0}\delta(\hat{\Omega}-\hat{\Omega}_0)$$
$$= F^S e^{-\tau/\mu_0}\delta(\mu-\mu_0)\delta(\phi-\phi_0)$$

- We have two sources to consider, the Sun and the rest of the medium

$$I^-(\tau,\mu,\phi) = I_d^-(\tau,\mu,\phi) + I_S^-(\tau,\mu,\phi)$$

Diffuse and direct components

- Assume (1) the lower surface is black, (2) no thermal radiation from the surface, then we can write the half range intensities as

$$-\mu \frac{dI^{-}(\tau,\hat{\Omega})}{d\tau} = I^{-}(\tau,\hat{\Omega}) - (1-a)B$$

$$-\frac{a}{4\pi}\int_{+} d\omega' \, p(+\hat{\Omega}',-\hat{\Omega}) I^{+}(\tau,\hat{\Omega}')$$

$$-\frac{a}{4\pi}\int_{-} d\omega' \, p(-\hat{\Omega}',-\hat{\Omega}) I^{-}(\tau,\hat{\Omega}')$$

Diffuse and direct components

- And for the + direction

$$-\mu \frac{dI^{+}(\tau,\hat{\Omega})}{d\tau} = I^{+}(\tau,\hat{\Omega}) - (1-a)B - \frac{a}{4\pi}\int_{+} d\omega'\, p(+\hat{\Omega}',+\hat{\Omega}) I^{+}(\tau,\hat{\Omega}') - \frac{a}{4\pi}\int_{-} d\omega'\, p(-\hat{\Omega}',+\hat{\Omega}) I^{-}(\tau,\hat{\Omega}')$$

Diffuse and direct components

Now substitute the sum of the direct and diffuse components

$$-\mu\frac{dI_d^-(\tau,\hat{\Omega})}{d\tau}-\mu\frac{dI_S^-(\tau,\hat{\Omega})}{d\tau}=I_d^-(\tau,\hat{\Omega})+I_S^-(\tau,\hat{\Omega})-(1-a)B$$

$$-\frac{a}{4\pi}\int_{-}d\omega'\,p(-\hat{\Omega}',-\hat{\Omega})I_S^-(\tau,\hat{\Omega}')-\frac{a}{4\pi}\int_{-}d\omega'\,p(+\hat{\Omega}',-\hat{\Omega})I_d^+(\tau,\hat{\Omega}')$$

$$-\frac{a}{4\pi}\int_{-}d\omega'\,p(-\hat{\Omega}',-\hat{\Omega})I_d^-(\tau,\hat{\Omega}')$$

Diffuse and direct components

But I_S^- is the direct solar beam and $dI_S^- = -I_S^- d\tau / \mu$

hence one gets

$$-\mu \frac{dI_d^-(\tau,\hat{\Omega})}{d\tau} = I_d^-(\tau,\hat{\Omega}) - (1-a)B - S^*(\tau,-\hat{\Omega})$$

$$-\frac{a}{4\pi}\int_+ d\omega' \, p(+\hat{\Omega}',-\hat{\Omega}) I_d^+(\tau,\hat{\Omega}')$$

$$-\frac{a}{4\pi}\int_- d\omega' \, p(-\hat{\Omega}',-\hat{\Omega}) I_d^-(\tau,\hat{\Omega}')$$

Note $I_d^-(\tau,\hat{\Omega}) = I_d(\tau,-\hat{\Omega})$

Diffuse and direct components

- where

$$S^*(\tau,-\hat{\Omega}) = \frac{a}{4\pi}\int_{-} d\omega'\, p(-\hat{\Omega}',-\hat{\Omega}) F^S e^{-\tau/\mu_0}\delta(\hat{\Omega}-\hat{\Omega}_0)$$

$$= \frac{a}{4\pi} p(-\hat{\Omega}_0,-\hat{\Omega}) F^S e^{-\tau/\mu_0}$$

- One can repeat this procedure for the upward component

Diffuse and direct components

$$-\mu \frac{dI_d^+(\tau,\hat{\Omega})}{d\tau} = I_d^+(\tau,\hat{\Omega}) - (1-a)B - S^*(\tau,+\hat{\Omega})$$

$$-\frac{a}{4\pi}\int_+ d\omega' \, p(+\hat{\Omega}',+\hat{\Omega}) I_d^+(\tau,\hat{\Omega}')$$

$$-\frac{a}{4\pi}\int_- d\omega' \, p(-\hat{\Omega}',+\hat{\Omega}) I_d^-(\tau,\hat{\Omega}')$$

Diffuse and direct components

$$S^*(\tau,+\hat{\Omega}) = \frac{a}{4\pi}\int_{-} d\omega'\, p(-\hat{\Omega}',+\hat{\Omega}) F^S e^{-\tau/\mu_0} \delta(\hat{\Omega}-\hat{\Omega}_0)$$

$$= \frac{a}{4\pi} p(-\hat{\Omega}_0,+\hat{\Omega}) F^S e^{-\tau/\mu_0}$$

Diffuse and direct components

- If we combine the half-range intensities we get

$$u\frac{dI(\tau,u,\phi)}{d\tau} = I(\tau,u,\phi) - \frac{a}{4\pi}\int_0^{2\pi} d\phi' \int_{-1}^{+1} du', p(u'.\phi';u,\phi)I(\tau,u',\phi')$$
$$-(1-a)B - S^*(\tau,u,\phi)$$

- Where *u* is cosθ and not |cosθ|